Symmetries, reduction, Hamilton-Jacobi theory and discretization for systems with external forces

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Introduction

Mechanical systems with external forces (also called non-conservative forces) are common in engineering, and can also arise when certain nonholonomic systems with symmetries are reduced.

In the Hamiltonian formalism, an external force is geometrically regarded as a semibasic 1-form α on T^*Q, i.e.,

$$\alpha = \alpha_i(q,p)\mathrm{d}q^i.$$

The dynamics of the forced Hamiltonian system (H,α) on T^*Q is given by the vector field $X_{H,\alpha}$, where

$$\iota_{X_{H,\alpha}}\omega_Q = \mathrm{d}H + \alpha,$$

where $\omega_Q = \mathrm{d}q^i \wedge \mathrm{d}p_i$ is the canonical symplectic structure on T^*Q.

Similarly, in the Lagrangian formalism an external force is regarded as a semibasic 1-form α on TQ. The dynamics of the forced Lagrangian system (L,α) is determined by the vector field $\xi_{L,\alpha}$, given by

$$\iota_{\xi_{L,\alpha}}\omega_L = \mathrm{d}E_L + \alpha,$$

where

$$\omega_L = \mathrm{d}q^i \wedge \mathrm{d}\left(\frac{\partial L}{\partial \dot q^i}\right), \quad E_L = \dot q^i \frac{\partial L}{\partial \dot q^i} - L, \quad \det\left(\frac{\partial^2 L}{\partial \dot q^i \dot q^j}\right) \neq 0.$$

Definition 1 *A **Rayleigh force** is an external force $\bar R$ on TQ given by*

$$\bar R = S^*(\mathrm{d}\mathcal{R}) = \frac{\partial \mathcal{R}}{\partial \dot q^i}\mathrm{d}q^i,$$

*where $\mathcal{R}$ is a function on TQ called the **Rayleigh potential**. A **Rayleigh system** $(L,\mathcal{R})$ is a forced Lagrangian system $(L,\bar R)$.*

Symmetries and constants of the motion

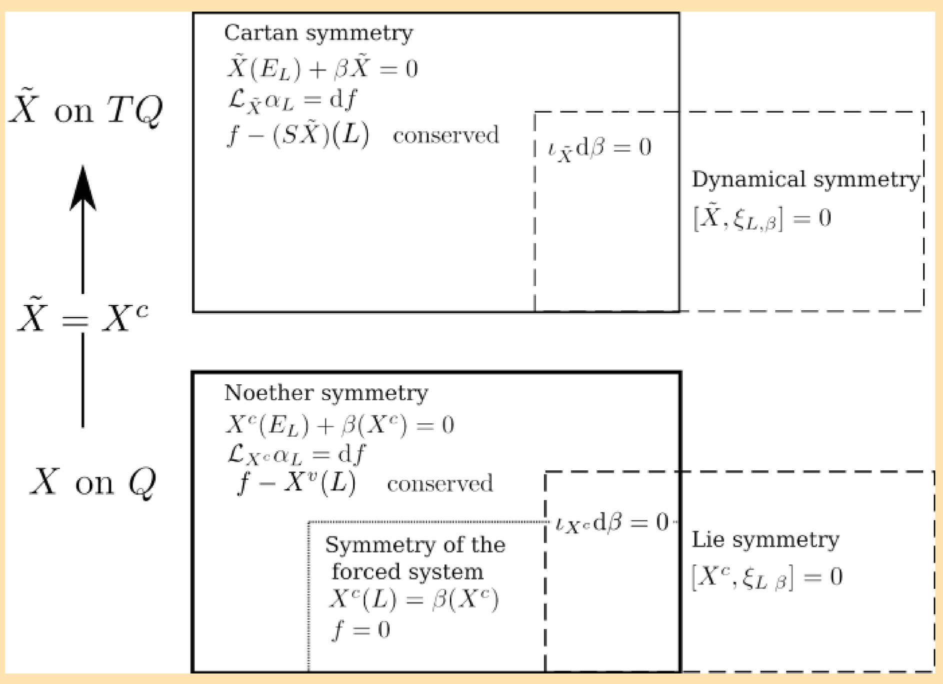


Fig. 1: Classification of symmetries and their conserved quantities

Example 1 (1D Rayleigh system) *Let*

$$L = \frac{1}{2}\dot q^2, \qquad \mathcal{R} = \frac{k}{3}\dot q^3; \qquad X = e^{kq}\frac{\partial}{\partial q}.$$

Then

$$X^c(L) = X^v(\mathcal{R}) \qquad \text{for} \qquad X = e^{kq}\frac{\partial}{\partial q},$$

so X is a symmetry of the forced Lagrangian and $X^v(L) = e^{kq}\dot q$ is a constant of the motion.

Reduction

Let G be a Lie group acting on Q and consider the lifted action to TQ by tangent prolongation. Assume the group action to be free and proper and let L be a G-invariant Lagrangian. The **momentum map** $J : TQ \to \mathfrak{g}^*$ is given by

$$J(\xi) = \theta_L(\xi_Q^c),$$

where $\theta_L = S^*(\mathrm{d}L) = \frac{\partial L}{\partial \dot q^i}\mathrm{d}q^i$. For each $\xi \in \mathfrak{g}$, we can define a function $J^\xi = J(\xi)$ on TQ.

Theorem 1 *Consider a $\mathfrak{g}$-invariant forced Lagrangian system (L,α) on TQ. Let $\mu \in \mathfrak{g}^*$, and w.l.o.g. assume that $Orb^{Ad^*}(\mu) = \{\mu\}$.*

i) The quotient space $(TQ)_\mu \coloneqq J^{-1}(\mu)/G$ is endowed with an induced symplectic structure ω_μ, given by

$$\pi_\mu^*\omega_\mu = \iota_\mu^*\omega_L,$$

where $\pi_\mu : J^{-1}(\mu) \to (TQ)_\mu$ and $\iota_\mu : J^{-1}(\mu) \hookrightarrow TQ$.

ii) The reduced Lagrangian L_μ and the reduced external forced α_μ are given by

$$L_\mu \circ \pi_\mu = L \circ \iota_\mu, \qquad \pi_\mu^*\alpha_\mu = \iota_\mu^*\alpha.$$

Hamilton-Jacobi theory

$$\begin{CD} T^*Q @>{X_{H,\beta}}>> TT^*Q \\ @A{\gamma}A{\pi_Q}A @VV{T\pi_Q}V \\ Q @>{X_{H,\beta}^\gamma}>> TQ \end{CD}$$

Theorem 2 *Let (H,β) be a forced Hamiltonian system on TQ. Let γ be a closed 1-form on Q. Then the following conditions are equivalent:*

i) $\mathrm{d}(H\circ\gamma) = -\gamma^*\beta$,

ii) if σ is an integral curve of $X_{H,\beta}^\gamma$, then $\gamma\circ\sigma$ is an integral curve of $X_{H,\beta}$,

iii) $\mathrm{Im}\,\gamma$ *is a Lagrangian submanifold of T^*Q and $X_{H,\beta}$ is tangent to it.*

*If γ satisfies these conditions, it is called a **solution of the Hamilton-Jacobi problem** for (H,β).*

A map $\Phi : Q\times\mathbb{R}^n \to T^*Q$ is called **complete solution of the Hamilton-Jacobi problem** for (H,β) if it is a diffeomorphism and $\Phi_\lambda(q) = \Phi(q,\lambda_1,\ldots,\lambda_n)$ is a solution of the Hamilton-Jacobi problem for (H,β).

Proposition 3 *The functions $f_a = \pi_a\circ\Phi^{-1} : T^*Q \to \mathbb{R}$ are constants of the motion. Moreover, they are in involution, i.e., $\{f_a, f_b\} = 0$.*

Example 2 *Consider a n-dimensional forced Hamiltonian system (H,β), with*

$$H = \frac{1}{2}\sum_{i=1}^n p_i^2, \qquad \beta = \sum_{i=1}^n \kappa_i p_i^2 \mathrm{d}q_i.$$

The functions $f_a = e^{\kappa_a q^a}p_a$, $a = 1,\ldots,n$ are constants of the motion in involution. The 1-form γ on Q given by

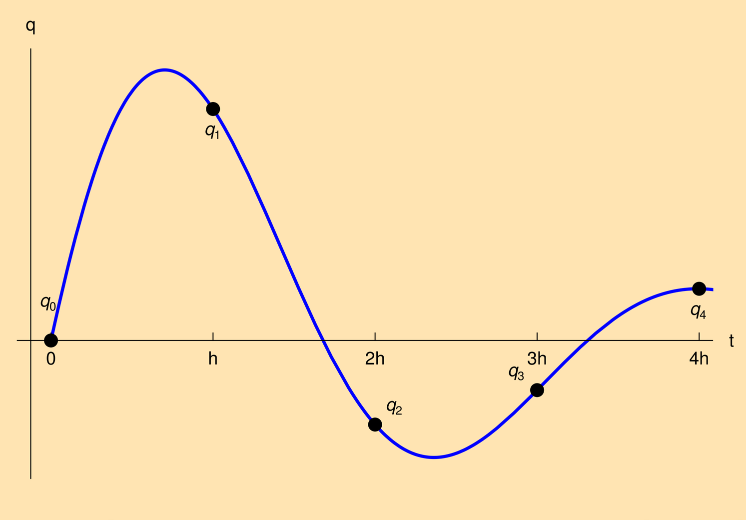
$$\gamma = \sum_{i=1}^n \lambda_i e^{-\kappa_i q^i}\mathrm{d}q^i$$

is a complete solution of the Hamilton-Jacobi problem.

Remark 1 *When H and β are G-invariant, we can reduce the Hamilton-Jacobi, find solutions on Q/G and reconstruct solutions on Q from them.*

Remark 2 *The Hamilton-Jacobi problem for a Čaplygin system can be reduced to the Hamilton-Jacobi problem for a forced Hamiltonian system without constraints.*

Discrete forced systems



In the discrete framework, TQ, L and α are replaced by $Q\times Q$, L_d and f_d, respectively. Discrete Lagrange-d'Alembert principle $\rightsquigarrow$ forced discrete Euler-Lagrange equations

Definition 2 *A **discrete Rayleigh force** $f_d = (f_d^-, f_d^+)$ is of the form*

$$f_d^-(q_0,q_1) = D_1\mathcal{R}(q_0,q_1), \quad f_d^+(q_0,q_1) = -D_2\mathcal{R}(q_0,q_1),$$

*where $\mathcal{R}_d$ is the so-called **discrete Rayleigh potential**.*

Example 3 (midpoint rule discretization) *Given a continuous Rayleigh potential $(L,\mathcal{R})$ on TQ, with $\partial\mathcal{R}/\partial q = 0$, we have*

$$L_d^{1/2}(q_0,q_1) = hL\left(q = \frac{q_0+q_1}{2}, \dot q = \frac{q_1-q_0}{h}\right), \quad \mathcal{R}_d^{1/2}(q_0,q_1) = \frac{h}{2}R\left(\dot q = \frac{q_1-q_0}{h}\right),$$

where h is the time step.

Remark 3 *We have developed a Hamilton-Jacobi theory for forced discrete systems.*

References

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