Liouville–Arnold theorem for contact Hamiltonian systems

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Introduction

- Roughly speaking, a completely integrable system is a Hamiltonian system with as much independent and "compatible" constants of the motion as degrees of freedom.
- A symplectic form ω on a manifold M is a 2-form on M such that $\mathrm{d}\omega = 0$ and $\omega(v, \cdot) = 0$ iff $v = 0$.
- Given a function f on M, its Hamiltonian vector field X_f is given by
$$\omega(X_f, \cdot) = \mathrm{d}f.$$
- The Poisson bracket $\{\cdot, \cdot\}$ is given by
$$\{f, g\} = \omega(X_f, X_g).$$

Theorem 1: Liouville–Arnold theorem

Let $f_1, \dots, f_n$ be independent functions in involution (i.e., $\{f_i, f_j\} = 0\ \forall i, j$) on a symplectic manifold (M^{2n}, ω). Let $M_\Lambda = \{x \in M \mid f_i = \Lambda_i\}$.

i) Any compact connected component of M_Λ is diffeomorphic to a torus $\mathbb{T}^n$.

ii) On a neighbourhood of M_Λ there are coordinates (φ^i, J_i) such that
$$\omega = \mathrm{d}\varphi^i \wedge \mathrm{d}J_i,$$
and the Hamiltonian dynamics are given by
$$\frac{\mathrm{d}\varphi^i}{\mathrm{d}t} = \Omega^i(J),$$
$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = 0.$$

Contact Hamiltonian systems

Definition 1

A (co-oriented) **contact manifold** is a pair (M, η), where M is an $(2n+1)$-dimensional manifold and η is a 1-form on M such that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form.

- There exists a unique vector field $\mathcal{R}$ on (M, η), called the **Reeb vector field**, such that
$$\mathrm{d}\eta(\mathcal{R}, \cdot) = 0, \qquad \eta(\mathcal{R}) = 1\,.$$
- The **Hamiltonian vector field** of $f \in C^\infty(M)$ is given by
$$\eta(X_f) = -f, \qquad \mathrm{d}\eta(X_f, \cdot) = \mathrm{d}f - \mathcal{R}(f)\eta\,.$$
- Around each point on M there exist **Darboux coordinates** (q^i, p_i, z) such that
$$\eta = \mathrm{d}z - p_i \mathrm{d}q^i, \qquad \mathcal{R} = \frac{\partial}{\partial z},$$
$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i \frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z}\,.$$
- The **Jacobi bracket** is given by
$$\{f, g\} = X_f(g) + g\mathcal{R}(f)\,.$$
- This bracket is bilinear and satisfies the Jacobi identity.
- However, unlike a Poisson bracket, it does not satisfy the Leibnitz identity:
$$\{f, gh\} \neq \{f, g\}h + \{f, h\}g.$$

Our approach

- Let (M, η) be a contact manifold with Jacobi bracket $\{\cdot, \cdot\}$.
- Consider $n+1$ independent functions $f_0, \dots, f_n \colon M \to \mathbb{R}$ in involution, i.e. $\{f_i, f_j\} = 0\ \forall\, i, j$.
- Then, $X_{f_i}(f_j) = f_j \mathcal{R}(f_i)$.
- Therefore, X_{f_i} in general is not tangent to $M_\Lambda = \{x \in M \mid f_\alpha = \Lambda_\alpha\}$.
- Other authors, such as Boyer and Jovanović, assume that $\mathcal{R}(f_\alpha) = 0\ \forall f_\alpha$ so that X_{f_i} is tangent to M_Λ.
- However, this leads to contact Hamiltonian dynamics without dissipation $\rightsquigarrow$ "symplectic dynamics".
- Instead of considering level sets M_Λ we consider preimages of rays:
$$M_{\langle\Lambda\rangle_+} = \{x \in M \mid \exists\, r \in \mathbb{R}^+\colon f_\alpha(x) = r\Lambda_\alpha\}\,.$$

Completely integrable contact systems

Theorem 2: Colombo, de León, Lainz, L.-G., 2023

Let (M, η) be a $(2n+1)$-dimensional contact manifold. Suppose that $f_0, f_1, \dots, f_n$ are functions in involution such that $(\mathrm{d}f_\alpha)$ has rank at least n. Then, $M_{\langle\Lambda\rangle_+}$ is invariant by the Hamiltonian flow of f_α and diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$.

Moreover, there is a neighborhood U of $M_{\langle\Lambda\rangle_+}$ such that

i) There exists coordinates $(y^0, \dots, y^n, \tilde{A}_1, \dots, \tilde{A}_n)$ on U, where $\tilde{A}_i = \tilde{A}_i(f_0, \dots, f_n)$, such that the equations of motion are given by
$$\frac{\mathrm{d}y^\alpha}{\mathrm{d}t} = \Omega^\alpha(\tilde{A}_i),$$
$$\frac{\mathrm{d}\tilde{A}_i}{\mathrm{d}t} = 0.$$

ii) There exists a conformal change $\tilde{\eta} = \eta / A_0$ such that $(y^i, \tilde{A}_i, y^0)$ are Darboux coordinates for $(M, \tilde{\eta})$, i.e. $\tilde{\eta} = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i$.

Definition 2

A **completely integrable contact system** is a triple (M, η, F), where (M, η) is a contact manifold and $F = (f_0, \dots, f_n)\colon M \to \mathbb{R}^{n+1}$ is a map such that the functions $f_0, \dots, f_n$ are in involution and $\mathrm{d}F$ has rank at least n on a dense open subset $M_0 \subseteq M$.

Example 1: Damped harmonic oscillator

- Let $M = \mathbb{R}^3$ with coordinates (q, p, z). Let $\eta = \mathrm{d}z - p\mathrm{d}q$.
- The functions $f = z - \frac{pq}{2}$ and $h = \frac{p^2}{2} + \frac{q^2}{2} + \kappa z$ are in involution. They are independent a.e.
- Hence, (M, η, F) is a completely integrable contact Hamiltonian system, where $F = (f, h)$.
- The integral curves of X_h,
$$\frac{\mathrm{d}q}{\mathrm{d}t} = p\,, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -q - \kappa p\,, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{p^2}{2} - \frac{q^2}{2} - \kappa z\,,$$
correspond to the dynamics of a harmonic oscillator with a linear damping:
$$\frac{\mathrm{d}^2 q}{\mathrm{d}t} = -q - \kappa \frac{\mathrm{d}q}{\mathrm{d}t}\,.$$

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