

# LIIOUVILLE–ARNOLD THEOREM FOR CONTACT HAMILTONIAN SYSTEMS

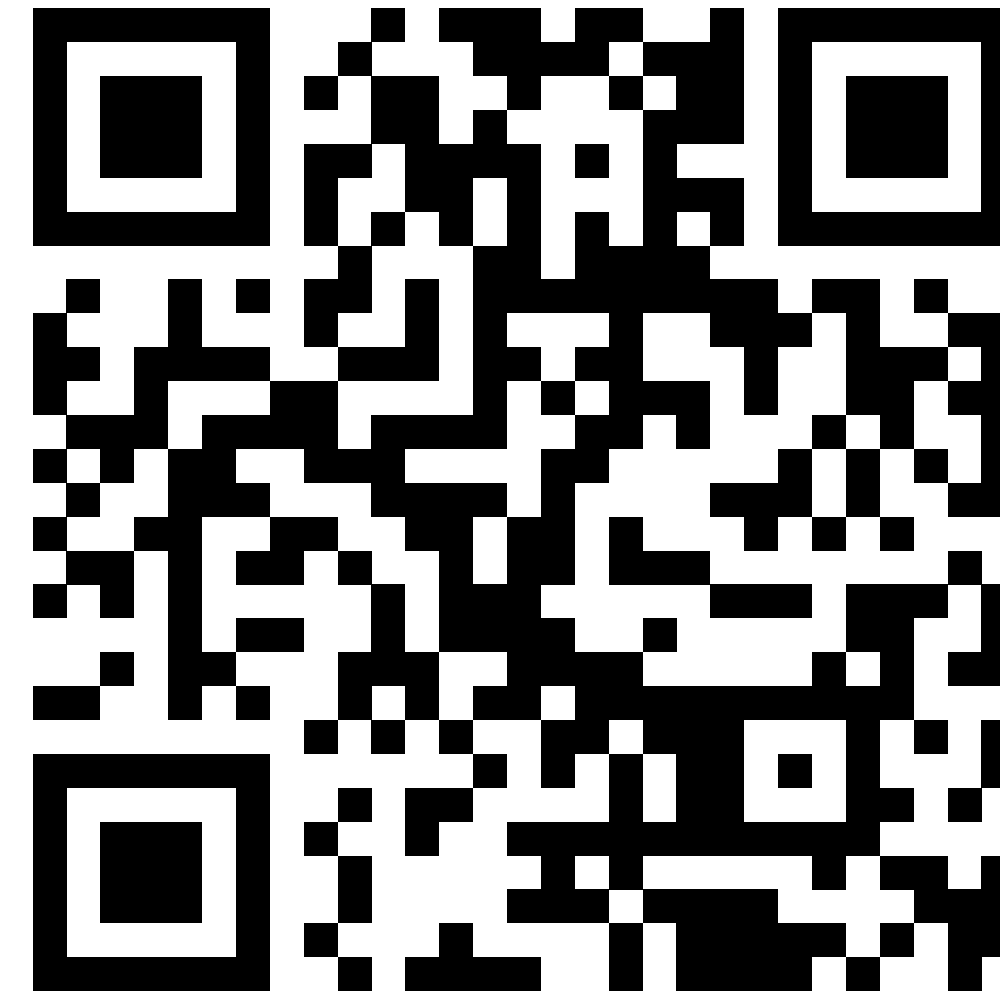
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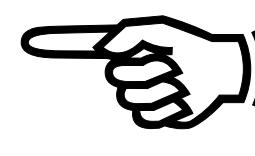
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## Introduction

- Roughly speaking, a completely integrable system is a Hamiltonian system with as much independent and “compatible” constants of the motion as degrees of freedom.
- A symplectic form  $\omega$  on a manifold  $M$  is a 2-form on  $M$  such that  $d\omega = 0$  and  $\omega(v, \cdot) = 0$  iff  $v = 0$ .
- Given a function  $f$  on  $M$ , its Hamiltonian vector field  $X_f$  is given by

$$\omega(X_f, \cdot) = df.$$

- The Poisson bracket  $\{\cdot, \cdot\}$  is given by

$$\{f, g\} = \omega(X_f, X_g).$$

### Theorem 1: Liouville–Arnold theorem

Let  $f_1, \dots, f_n$  be independent functions in involution (i.e.,  $\{f_i, f_j\} = 0 \forall i, j$ ) on a symplectic manifold  $(M^{2n}, \omega)$ . Let  $M_\Lambda = \{x \in M \mid f_i = \Lambda_i\}$ .

- Any compact connected component of  $M_\Lambda$  is diffeomorphic to a torus  $\mathbb{T}^n$ .
- On a neighbourhood of  $M_\Lambda$  there are coordinates  $(\varphi^i, J_i)$  such that

$$\omega = d\varphi^i \wedge dJ_i,$$

and the Hamiltonian dynamics are given by

$$\frac{d\varphi^i}{dt} = \Omega^i(J),$$

$$\frac{dJ_i}{dt} = 0.$$

## Our approach

- Let  $(M, \eta)$  be a contact manifold with Jacobi bracket  $\{\cdot, \cdot\}$ .
- Consider  $n + 1$  independent functions  $f_0, \dots, f_n: M \rightarrow \mathbb{R}$  in involution, i.e.  $\{f_i, f_j\} = 0 \forall i, j$ .
- Then,  $X_{f_i}(f_j) = f_j \mathcal{R}(f_i)$ .
- Therefore,  $X_{f_i}$  in general is not tangent to  $M_\Lambda = \{x \in M \mid f_\alpha = \Lambda_\alpha\}$ .
- Other authors, such as Boyer and Jovanović, assume that  $\mathcal{R}(f_\alpha) = 0 \forall f_\alpha$  so that  $X_{f_i}$  is tangent to  $M_\Lambda$ .
- However, this leads to contact Hamiltonian dynamics without dissipation  $\rightsquigarrow$  “symplectic dynamics”.
- Instead of considering level sets  $M_\Lambda$  we consider preimages of rays:

$$M_{(\Lambda)_+} = \{x \in M \mid \exists r \in \mathbb{R}^+ : f_\alpha(x) = r\Lambda_\alpha\}.$$

## Completely integrable contact systems

### Theorem 2: Colombo, de León, Lainz, L.-G., 2023

Let  $(M, \eta)$  be a  $(2n + 1)$ -dimensional contact manifold. Suppose that  $f_0, f_1, \dots, f_n$  are functions in involution such that  $(df_\alpha)$  has rank at least  $n$ . Then,  $M_{(\Lambda)_+}$  is invariant by the Hamiltonian flow of  $f_\alpha$  and diffeomorphic to  $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$ .

Moreover, there is a neighborhood  $U$  of  $M_{(\Lambda)_+}$  such that

- There exists coordinates  $(y^0, \dots, y^n, \tilde{A}_1, \dots, \tilde{A}_n)$  on  $U$ , where  $\tilde{A}_i = \tilde{A}_i(f_0, \dots, f_n)$ , such that the equations of motion are given by

$$\frac{dy^\alpha}{dt} = \Omega^\alpha(\tilde{A}_i),$$

$$\frac{d\tilde{A}_i}{dt} = 0.$$

- There exists a conformal change  $\tilde{\eta} = \eta/A_0$  such that  $(y^i, \tilde{A}_i, y^0)$  are Darboux coordinates for  $(M, \tilde{\eta})$ , i.e.  $\tilde{\eta} = dy^0 - \tilde{A}_i dy^i$ .

## Contact Hamiltonian systems

### Definition 1

A (co-oriented) **contact manifold** is a pair  $(M, \eta)$ , where  $M$  is an  $(2n + 1)$ -dimensional manifold and  $\eta$  is a 1-form on  $M$  such that  $\eta \wedge (d\eta)^n$  is a volume form.

- There exists a unique vector field  $\mathcal{R}$  on  $(M, \eta)$ , called the **Reeb vector field**, such that

$$d\eta(\mathcal{R}, \cdot) = 0, \quad \eta(\mathcal{R}) = 1.$$

- The **Hamiltonian vector field** of  $f \in C^\infty(M)$  is given by

$$\eta(X_f) = -f, \quad d\eta(X_f, \cdot) = df - \mathcal{R}(f)\eta.$$

- Around each point on  $M$  there exist **Darboux coordinates**  $(q^i, p_i, z)$  such that

$$\eta = dz - p_i dq^i, \quad \mathcal{R} = \frac{\partial}{\partial z},$$

$$X_f = \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \left( \frac{\partial f}{\partial q^i} + p_i \frac{\partial f}{\partial z} \right) \frac{\partial}{\partial p_i} + \left( p_i \frac{\partial f}{\partial p_i} - f \right) \frac{\partial}{\partial z}.$$

- The **Jacobi bracket** is given by

$$\{f, g\} = X_f(g) + g\mathcal{R}(f).$$

- This bracket is bilinear and satisfies the Jacobi identity.

- However, unlike a Poisson bracket, it does not satisfy the Leibnitz identity:

$$\{f, gh\} \neq \{f, g\}h + \{f, h\}g.$$

### Definition 2

A **completely integrable contact system** is a triple  $(M, \eta, F)$ , where  $(M, \eta)$  is a contact manifold and  $F = (f_0, \dots, f_n): M \rightarrow \mathbb{R}^{n+1}$  is a map such that the functions  $f_0, \dots, f_n$  are in involution and  $dF$  has rank at least  $n$  on a dense open subset  $M_0 \subseteq M$ .

### Example 1: Damped harmonic oscillator

- Let  $M = \mathbb{R}^3$  with coordinates  $(q, p, z)$ . Let  $\eta = dz - pdq$ .
- The functions  $f = z - \frac{pq}{2}$  and  $h = \frac{p^2}{2} + \frac{q^2}{2} + \kappa z$  are in involution. They are independent a.e.
- Hence,  $(M, \eta, F)$  is a completely integrable contact Hamiltonian system, where  $F = (f, h)$ .
- The integral curves of  $X_h$ ,

$$\frac{dq}{dt} = p, \quad \frac{dp}{dt} = -q - \kappa p, \quad \frac{dz}{dt} = \frac{p^2}{2} - \frac{q^2}{2} - \kappa z,$$

correspond to the dynamics of a harmonic oscillator with a linear damping:

$$\frac{d^2q}{dt^2} = -q - \kappa \frac{dq}{dt}.$$

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