The geometry of Rayleigh dissipation

Symmetries, constants of the motion and reduction of systems with external forces

Asier López Gordón

Instituto de Ciencias Matemáticas
(CSIC-UAM-UC3M-UCM)
Supervisor: Manuel de León
Tutor: Germán Sierra

July 5, 2021

Universidad Autónoma de Madrid

INSTITUTO DE CIENCIAS MATEMÁTICAS

Symplectic geometry I

- Recall that a p-**form** is a $(0, p)$-type skew-symmetric tensor field
- A 2-form ω is called **symplectic** if:
  1. it is closed ($\mathrm{d}\omega{=}0$),
  2. it is non-degenerate ($\omega(X, Y) = 0\ \forall Y \in TM \Rightarrow X = 0$).
- A **symplectic manifold** (M, ω) is an $2m$-dimensional manifold M endowed with a symplectic form ω.
- The tautological 1-form on T^*Q is given by

$$\theta = p_i \mathrm{d}q^i,$$

and the canonical symplectic form is

$$\omega = -\mathrm{d}\theta = \mathrm{d}q^i \wedge \mathrm{d}p_i.$$

Symplectic geometry II

- The **interior product** of a vector field X by a p-form α is the $p-1$ form $\iota_X\alpha$ such that

$$(\iota_X\alpha)(Y_1,\ldots,Y_{p-1}) = \alpha(X, Y_1,\ldots,Y_{p-1}).$$

In particular, if α is a 1-form,

$$\iota_X\alpha = \alpha(X)$$

- Consider a differentiable function $f : M \to \mathbb{R}$ on (M,ω). The **Hamiltonian vector field** X_f on (M,ω) is given by

$$\iota_{X_f}\omega = \mathrm{d}f.$$

Hamilton's equations in its intrinsic form

- The trajectories of the system are the integral curves of the vector field X_H on (T^*Q, ω), given by

$$\iota_{X_H}\omega = \mathrm{d}H.$$

- In bundle coordinates (q^i, p_i), this implies that

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}.$$

- The integral curves of X_H are then given by

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i},$$
$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}.$$

Forced Hamilton's equations

- Geometrically, an external force is represented by a semibasic 1-form γ on T^*Q. Locally,
$$\gamma = \gamma_i(q,p)\,\mathrm{d}q^i.$$
- The dynamic vector field is now $X_{H,\gamma}$, given by
$$\iota_{X_{H,\gamma}}\omega = \mathrm{d}H + \gamma.$$
- The equations of motion are then
$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i},$$
$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\left(\frac{\partial H}{\partial q^i} + \gamma_i\right).$$

Vertical and complete lifts of a vector field

- If Q has coordinates (q^i), it induces coordinates $(q^i, \dot{q}^i)$ on TQ.
- Consider a vector field X on Q locally given by

$$X = X^i \frac{\partial}{\partial q^i}.$$

- Its **vertical lift** is the vector field X^v on TQ given by

$$X^v = X^i \frac{\partial}{\partial \dot{q}^i}.$$

- Its **complete lift** is the vector field X^c on TQ given by

$$X^c = X^i \frac{\partial}{\partial q^i} + \dot{q}^j \frac{\partial X^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i}.$$

Canonical lift of a curve

- Consider the curve σ on Q given by

$$\begin{aligned}\sigma : I \subset \mathbb{R} &\to Q\\ t &\mapsto (q^i(t)).\end{aligned}$$

- Its **canonical lift** is the curve $\tilde{\sigma}$ on TQ given by

$$\begin{aligned}\tilde{\sigma} : I \subset \mathbb{R} &\to TQ\\ t &\mapsto (q^i(t), \dot{q}^i(t)).\end{aligned}$$

- For instance, if $\sigma(t) = (\cos(\omega t), \sin(\omega t))$, then

$$\tilde{\sigma}(t) = (\cos(\omega t), \sin(\omega t), -\omega \sin(\omega t), \omega \cos(\omega t)).$$

SODE

- A **second order differential equation** (**SODE**) is a vector field ξ on TQ of the form
$$\xi = \dot{q}^i \frac{\partial}{\partial q^i} + \xi^i(q^i, \dot{q}^i) \frac{\partial}{\partial \dot{q}^i}.$$
- A **solution** of a SODE ξ is a curve $\sigma(t) = (q^i(t))$ on Q such that $\tilde{\sigma}$ is an integral curve of ξ, given by the second order differential equations
$$\frac{\mathrm{d}^2 q^i}{\mathrm{d}t^2} = \xi^i\left(q^i, \frac{\mathrm{d}q^i}{\mathrm{d}t}\right), \quad 1 \leq i \leq n = \dim Q.$$

Symplectic structure on TQ induced by the Lagrangian

- Consider a Lagrangian function L on TQ.
- The Poincaré-Cartan forms are given by

$$\alpha_L = \frac{\partial L}{\partial \dot{q}^i} \mathrm{d}q^i,$$

$$\omega_L = -\mathrm{d}\alpha_L = \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} \mathrm{d}q^i \wedge \mathrm{d}q^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \mathrm{d}q^i \wedge \mathrm{d}\dot{q}^j.$$

- L is called **regular** if ω_L is symplectic.

Euler-Lagrange equations in its intrinsic form

- The energy of the system is

$$E_L = \Delta(L) - L, \qquad \Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}.$$

- The dynamics of the system is determined by the **Euler-Lagrange vector field** ξ_L, given by

$$\iota_{\xi_L} \omega_L = \mathrm{d} E_L$$

- ξ_L is a SODE and its solutions satisfy the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad 1 \leq i \leq n.$$

Forced Euler-Lagrange equations

- An external force is represented by a semibasic 1-form β on TQ. Locally,
$$\beta = \beta_i(q, \dot{q}) \, \mathrm{d}q^i .$$
- The dynamics is now determined by the **forced Euler-Lagrange vector field** $\xi_{L,\beta}$, given by
$$\iota_{\xi_{L,\beta}} \omega_L = \mathrm{d}E_L + \beta .$$
- $\xi_{L,\beta}$ is also a SODE, with solutions given by
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = -\beta_i, \quad 1 \leq i \leq n .$$

Rayleigh's hypothesis

- Rayleigh considered an external force which is linear in the velocities:

$$\bar{R} = R_{ij}(q)\dot{q}^i \mathrm{d}q^j, \qquad R_{ij} = R_{ji}.$$

- We can introduce the **Rayleigh dissipation function**:

$$\mathcal{R} = \frac{1}{2} R_{ij}(q)\dot{q}^i \dot{q}^j,$$

so that

$$\bar{R} = \frac{\partial \mathcal{R}}{\partial \dot{q}^i} \mathrm{d}q^i.$$

Generalized Rayleigh dissipation

- We can consider a "potential" $\mathcal{R}$ on TQ (not necessarily quadratic in the velocities), from which an external force is derived.
- $\mathcal{R}$ expresses the energy dissipated away by the system:

$$\frac{\mathrm{d}}{\mathrm{d}t} E_L \circ \sigma(t) = -\Delta(\mathcal{R}) \circ \sigma(t),$$

with σ an integral curve of $\xi_{L,\bar{R}}$.

Noether theorem

Theorem (Noether's theorem for forced Lagrangian systems)

Let X be a vector field on Q. Then $X^c(L) = \beta(X^c)$ if and only if $X^v(L)$ is a constant of the motion.

- A vector field X on Q satisfying these conditions is called a **symmetry of the forced Lagrangian** (L, β).
- For a Rayleigh system $(L, \mathcal{R})$, this is equivalent to

$$X^c(L) = X^v(\mathcal{R}).$$

Other point-like symmetries I

- We want to consider other infinitesimal transformations on Q that leave the system (L, β) invariant.
- A **Lie symmetry** is a vector field X on Q such that

$$[X^c, \xi_{L,\beta}] = \mathcal{L}_{X^c} \xi_{L,\beta} = 0$$

- A **Noether symmetry** is a vector field X on Q such that

$$\mathcal{L}_{X^c} \alpha_L = \mathrm{d} f, \qquad X^c(E_L) + \beta(X^c) = 0.$$

- If $\mathcal{L}_{X^c} \alpha_L = \mathrm{d} f$, then X is a Noether symmetry if and only if $f - X^v(L)$ is a conserved quantity.

Other point-like symmetries II

- For a Rayleigh system $(L, \mathcal{R})$, if $\mathcal{L}_{X^c} \alpha_L = \mathrm{d} f$, then X is a Noether symmetry if and only if

$$X^c(E_L) + X^v(\mathcal{R}) = 0.$$

- If X is a Noether symmetry, it is also a symmetry of the forced Lagrangian if and only if $\mathcal{L}_{X^c} \alpha_L = 0$.
- If X is a Noether symmetry, it is also a Lie symmetry if and only if

$$\iota_{X^c} \mathrm{d}\beta = 0.$$

Non-point-like symmetries I

- We want to consider infinitesimal transformations on TQ that leave the system (L, β) invariant.
- A vector field $\tilde{X}$ on TQ is called a **dynamical symmetry** if

$$[\tilde{X}, \xi_{L,\beta}] = 0.$$

- A vector field $\tilde{X}$ on TQ is called a **Cartan symmetry** if

$$\mathcal{L}_{\tilde{X}} \alpha_L = \mathrm{d} f, \qquad \tilde{X}(E_L) + \beta(\tilde{X}) = 0$$

- X is a Lie symmetry if and only if X^c is a dynamical symmetry.
- X is a Noether symmetry if and only if X^c is a Cartan symmetry.

Non-point-like symmetries II

- If $\mathcal{L}_{\tilde{X}}\alpha_L$ is closed, then $\tilde{X}$ is a dynamical symmetry if and only if

$$\mathrm{d}(\tilde{X}(E_L)) = -\mathcal{L}_{\tilde{X}}\beta.$$

- A Cartan symmetry is a dynamical symmetry if and only if

$$\iota_{\tilde{X}}\mathrm{d}\beta = 0.$$

- If $\mathcal{L}_{\tilde{X}}\alpha_L = \mathrm{d}f$, then $\tilde{X}$ is a Cartan symmetry if and only if $f - (S\tilde{X})(L)$ is a constant of the motion.
- For a Rayleigh system $(L, \mathcal{R})$, $\tilde{X}$ is a Cartan symmetry if and only if

$$\tilde{X}(E_L) + (S\tilde{X})(\mathcal{R}) = 0.$$

Momentum map

- Consider a G-invariant regular Lagrangian L on TQ, where G is a Lie group with Lie algebra $\mathfrak{g}$ and dual Lie algebra $\mathfrak{g}^*$.
- The **natural momentum map** is given by

$$
\begin{aligned}
J : TQ &\to \mathfrak{g}^* \\
\langle J(x), \xi \rangle &= \alpha_L(\xi_Q^c)
\end{aligned}
$$

for each $\xi \in \mathfrak{g}$.

- For each $\xi \in \mathfrak{g}$,

$$
\begin{aligned}
J^\xi : TQ &\to \mathbb{R} \\
x &\mapsto \langle J(x), \xi \rangle
\end{aligned}
$$

is a function on TQ.

Group actions and quotient manifold

- **Idea**: reducing the dimensions of TQ (i.e., taking out redundant d.o.f.) when L is G-invariant.
- The group action $\Phi : G \times M \to M$ on a manifold M needs to be
  1. **free**: for every $x \in M$, $\Phi_g(x) = x$ if and only if $g = \mathrm{id}_G$,
  2. **proper**: for any compact subset $K \subset M$, $\Phi^{-1}(K)$ is also compact.
- Equivalence relation: $x \sim y$ if $\exists\ g \in G$ such that $\Phi(g, x) = y$.
- The orbit of x and the orbit space are

$$[x] = \{y \in M \mid y \sim x\}, \qquad M/G = \{[x] \mid x \in M\},$$

  respectively,
- Φ smooth, free and proper $\implies$ M/G is a differentiable manifold of dimension $\dim M - \dim G$.

Lemma

Consider a forced Lagrangian system (L, β). *Let* $\xi \in \mathfrak{g}$. *Then*

1. J^ξ *is a conserved quantity if and only if*

$$\beta(\xi_Q^c) = 0.$$

2. *If the previous equation holds, then* ξ *leaves* β *invariant if and only if*

$$\iota_{\xi_Q^c} \mathrm{d}\beta = 0.$$

In addition, the vector subspace of $\mathfrak{g}$ *given by*

$$\mathfrak{g}_\beta = \left\{ \xi \in \mathfrak{g} \mid \beta(\xi_Q^c) = 0,\ \iota_{\xi_Q^c} \mathrm{d}\beta = 0 \right\}$$

is a Lie subalgebra of $\mathfrak{g}$.

Isotropy group

- J^ξ is a constant of the motion $\forall \xi \in \mathfrak{g}_\beta \Rightarrow J_\beta^{-1}(\mu)$ is left invariant by the flow of $\xi_{L,\beta}$.
- Therefore the integral curves of $\xi_{L,\beta}$ are contained in level sets $J_\beta^{-1}(\mu) \subset TQ$.
- The isotropy Lie algebra at $\mu \in \mathfrak{g}_\beta^*$ is

$$(\mathfrak{g}_\beta)_\mu = \{\xi \in \mathfrak{g}_\beta \mid \langle \mu, [\xi, \eta] \rangle = 0 \ \forall \eta \in \mathfrak{g}_\beta\}.$$

- $(G_\beta)_\mu \leq G$ is the Lie group generated by $(\mathfrak{g}_\beta)_\mu$.

Theorem

Consider a $\mathfrak{g}_\beta$*-invariant forced Lagrangian system* (L, β) *on* TQ*. Let* $\mu \in \mathfrak{g}_\beta^*$*. Then:*

1. *The quotient space* $(TQ)_\mu := J_\beta^{-1}(\mu)/(G_\beta)_\mu$ *is endowed with an induced symplectic structure* ω_μ*, given by*

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega_L,$$

   where $\pi_\mu : J_\beta^{-1}(\mu) \to (TQ)_\mu$ *and* $i_\mu : J_\beta^{-1}(\mu) \hookrightarrow TQ$*.*

2. *The reduced Lagrangian* L_μ *is given by*

$$L_\mu \circ \pi_\mu = L \circ i_\mu.$$

3. *The reduced external force* β_μ *is given by*

$$\pi_\mu^* \beta_\mu = i_\mu^* \beta.$$

Main results

1. Generalization of Noether's theorem for forced Lagrangian systems.
2. Study and classification of symmetries for forced mechanical systems.
3. Development of a reduction theory for forced Lagrangian systems.
4. Geometric description of Rayleigh forces, particularizing the results above.

Future work

1. Extension of the results to higher-order systems: $L(q, \dot{q}, \ddot{q}, \ldots, q^{(k)})$ on $T^k Q$.
2. Extension of the results to non-autonomous systems: $L(q, \dot{q}, t)$ on $TQ \times \mathbb{R}$.
3. Hamilton-Jacobi theory:

$$H\left(q^i, \frac{\partial W}{\partial q^i}\right) = E \Leftrightarrow (\mathrm{d}W^*)H = E$$

4. Discrete mechanics and geometric integrators.

Main references

R. Abraham and J. Marsden, *Foundations of Mechanics*. AMS Chelsea Pub., 2008, ISBN: 978-0-8218-4438-0.

M. de León, M. Lainz, and A. López-Gordón, *Geometric Hamilton-Jacobi theory for systems with external forces and discretization*, Under preparation, 2021.

——,“Symmetries, constants of the motion, and reduction of mechanical systems with external forces,” *J. Math. Phys.*, vol. 62, no. 4, p. 042 901, 2021.

M. de León and P. R. Rodrigues, *Methods of differential geometry in analytical mechanics*. North-Holland, 1989, ISBN: 978-0-444-88017-8.

Thanks for your attention!