An introduction to integrable systems

Asier López-Gordón

Instituto de Ciencias Matemáticas (ICMAT-CSIC), Madrid (Spain)

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Outline of the presentation

1. Introduction
2. Liouville integrability
3. Hamilton–Jacobi theory
4. KAM theory
5. Generalizations

Introduction

- Roughly speaking, a completely integrable system is a mechanical system with n independent and "compatible" constants of the motion, where n is the number of degrees of freedom.
- In such systems, the equations of motion can be completely "solved", being reduced to quadratures.

Preliminary concepts

- Let (M, ω) be a symplectic manifold.
- Recall that the Poisson bracket $\{\cdot, \cdot\}\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is given by
$$\{f, g\} = \omega(X_f, X_g).$$

Definition

A collection of functions $f_1, \ldots, f_n \in C^\infty(M)$ are said to be **in involution** if $\{f_i, f_j\} = 0$ for each $i, j = 1, \ldots, n$.

Proposition

Let (M, ω, h) be a Hamiltonian system. A function $f \in C^\infty(M)$ is a conserved quantity iff it is in involution with h.

Preliminary concepts

Definition

A submanifold $N \subset M$ of a symplectic manifold (M^{2n}, ω) is called **Lagrangian** if $\dim N = n$ and $\omega|_N = 0$.

Proposition

Let T^*Q *be the cotangent bundle of* Q *and let* θ *denote its tautological one-form. A one-form* $\alpha \in \Omega^1(Q)$ *is closed iff* $\operatorname{Im} \alpha$ *is a Lagrangian submanifold of* $(\mathrm{T}^*Q, \mathrm{d}\theta)$.

Definition

A Hamiltonian system (M, ω, h) is called **completely integrable** (or **Liouville integrable**) if there exists n functions $f_1, f_2, \ldots, f_n \in C^\infty(M)$ such that

1. $h, f_1, f_2, \ldots, f_n$ are in involution,
2. they are functionally independent (i.e. $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \neq 0$) almost everywhere,

The functions $f_1, f_2, \ldots, f_n$ are called **integrals**.

Theorem (Liouville–Arnold theorem)

Let (M, ω, h) *be a completely integrable system. Let* M_Λ *be a regular level set of the integrals* $f_1, \ldots, f_n$*, i.e.*

$$M_\Lambda = \{x \in M \mid f_i = \Lambda_i\}, \quad \mathrm{d}_x f_1 \wedge \cdots \wedge \mathrm{d}_x f_n \neq 0 \ \forall\, x \in M_\Lambda\,.$$

Then

1. M_Λ *is a Lagrangian submanifold of* (M, ω)*.*
2. M_Λ *is invariant w.r.t. the flow of* X_h *and* X_{f_i}*.*
3. *Any compact connected component of* M_Λ *is diffeomorphic to* $\mathbb{T}^n$*.*
4. *On a neighborhood of* M_Λ *there are coordinates* (φ^i, s_i) *such that*
   - A. $\omega = \mathrm{d}\varphi^i \wedge \mathrm{d}s_i$,
   - B. *the action coordinates* s_i *are functions of the integrals* $f_1, \ldots, f_n$*,*
   - C. *the integral curves of* X_h *are given by*

$$\dot{\varphi}^i = \Omega^i(s_1, \ldots, s_n), \qquad \dot{s}_i = 0\,.$$

Proof of ① and ②

- Since $X_{f_j} f_i = \{f_i, f_j\} = 0 \ \forall\, i, j$ and $T_x M_\Lambda = \ker\{\mathrm{d} f_i\}$, the vector fields X_{f_i} are tangent to M_Λ. In other words, their flows leave M_Λ invariant.
- Since $\mathrm{d} f_1, \ldots, \mathrm{d} f_n$ are linearly independent and $v \mapsto \iota_v \omega$ is an isomorphism, $X_{f_1}, \ldots, X_{f_n}$ are linearly independent.
- Hence, $\{X_{f_1}(x), \ldots, X_{f_n}(x)\}$ is a basis of $T_x M_\Lambda$ for each $x \in N$.
- Using that $\omega(X_{f_i}, X_{f_j}) = \{f_i, f_j\} = 0$ for each i, j, we conclude that M_Λ is Lagrangian.

Sketch of the proof of ③

Lemma

Let N be an n-dimensional connected manifold and let $X_1, \ldots, X_n \in \mathfrak{X}(N)$ be linearly independent complete vector fields. If these vector fields are pairwise commutative, then N is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$ for some $k \leq n$. In particular, if N is compact then $N \simeq \mathbb{T}^n$.

- Under the hypotheses of the lemma, the flows of X_i define a Lie group action $\Phi \colon \mathbb{R}^n \times N \to M$ of R^n on M given by

$$\Phi(t_1, \ldots, t_n)(x) = \phi_{t_1}^{X_1} \circ \cdots \circ \phi_{t_n}^{X_n}(x) .$$

- Since X_i are linearly independent, for each $x \in N$ the mapping $A_x \colon (t_1, \ldots, t_n) \mapsto \Phi(t_1, \ldots, t_n)(x)$ is an immersion.

Sketch of the proof of ③

- Hence, $O(x) = \operatorname{Im} A_x$ is open in N.
- Since N is connected, $O(x) = N$.
- Every orbit $O(x)$ is the quotient space of $\mathbb{R}^n$ by the isotropy subgroup G_x.
- Since A_x is a local diffeomorphism, G_x is discrete.
- One can show that G_x is a lattice $\mathbb{Z}^k$ for some $k \leq n$.
- We conclude that $N \simeq \mathbb{R}^n/\mathbb{Z}^k \simeq \mathbb{T}^k \times \mathbb{R}^{n-k}$.

Sketch of the proof of ④

- Let $F = (f_1, \dots, f_n) \colon M \to \mathbb{R}^n$.
- For each point $y \in \mathbb{R}^n$, there exists a neighborhood D of a such that $F^{-1}(D)$ is diffeomorphic to $D \times F^{-1}(y)$. Moreover, $F \colon D \times F^{-1}(y) \to D$ is a trivial bundle.
- Consider a neighborhood $U = D \times M_\Lambda$ of $M_\Lambda = F^{-1}(\Lambda)$ as above.
- The integrals $f_1, \dots, f_n$ are coordinates on D.
- Let $(\psi^1, \dots, \psi^n)$ be angular coordinates of the torus $\mathbb{T}^n \simeq M_\Lambda$.
- The integrals $f_1, \dots, f_n$ are coordinates on D.
- In coordinates (ψ^i, f_i), the symplectic form is given by

$$\omega = a_{ij} \mathrm{d}\psi^i \wedge \mathrm{d}\psi^j + c_i^j \mathrm{d}\psi^i \wedge \mathrm{d}f_j + b^{ij} \mathrm{d}f_i \wedge \mathrm{d}f_j \,.$$

- Since M_Λ is Lagrangian, $0 = \omega|_{M_\Lambda} = a_{ij} \mathrm{d}\psi^i \wedge \mathrm{d}\psi^j$.

Sketch of the proof of ④

- One can show that c_i^j and b^{ij} do not depend on $\psi^1, \ldots, \psi^n$.
- Hence,

$$\omega = \mathrm{d}\psi^i \wedge \underbrace{\left(c_i^j \mathrm{d}f_j\right)}_{\omega_i} + \underbrace{b^{ij}\mathrm{d}f_i \wedge \mathrm{d}f_j}_{\beta} = \mathrm{d}\psi^i \wedge \omega_i + \beta\,.$$

- Since they do not depend on (ψ^i), $\omega_i \in \Omega^1(D)$, $\beta \in \Omega^2(D)$.
- The fact that ω is closed implies that ω_i and β are closed.
- Thus they are exact, namely, $\omega_i = \mathrm{d}s_i$ and $\beta = \mathrm{d}\gamma$.
- The functions $s_1 = s_1(f_1, \ldots, f_n), \ldots, s_n = s_n(f_1, \ldots, f_n)$ are independent, so (ψ^i, s_i) are coordinates on U.
- Let $\varphi^i = \psi^i + \gamma^i(s_1, \ldots s_n)$, where $\gamma = \gamma^i \mathrm{d}s_i$.

Sketch of the proof of ④

- Geometrically, this means that we change the initial points of reference for the angle coordinates on the torus.
- Hence,

$$\mathrm{d}\varphi^i \wedge \mathrm{d}s_i = \mathrm{d}\psi^i \wedge \mathrm{d}s_i + \mathrm{d}\gamma = \omega \,.$$

- Observe that $\frac{\partial}{\partial \varphi^i} = X_{s_i}$, so

$$\frac{\partial h}{\partial \varphi^i} = X_{s_i}(h) = \{s_i(f_1, \ldots, f_n), h\} = 0 \,,$$

and we have $h = f(s_1, \ldots, s_n)$.

Sketch of the proof of (4)

- Moreover,

$$X_h = \underbrace{\frac{\partial h}{\partial s_i}}_{\Omega^i} \frac{\partial}{\partial \varphi^i}\,,$$

where the frequencies Ω^i depend only on the action coordinates.

Explicit expression for angle variables

- Choose a point $x \in M_\Lambda$ and consider the solutions of the equation

$$\Phi(e_i)(x) = x\,.$$

- Then $\{e_1, \ldots e_n\}$ is a basis of the lattice $\mathbb{Z}^n$.
- By the implicit function theorem, e_i will depend on x smoothly.
- If $y = \Phi(a)x$, where $a = a^1 e_1 + \cdots + a^n e_n \in \mathbb{R}^n$, define the angle coordinates by

$$\psi^i = 2\pi a^i \pmod{2\pi}\,.$$

Explicit expression for action variables

- Fixing the basis $\{e_1, \dots e_n\}$ uniquely determines the set of basis cycles $\gamma_1, \dots, \gamma_n$ in the fundamental group $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$.
- Let $\alpha \in \Omega^1(U)$ such that $\mathrm{d}\alpha = \omega$.
- To each torus, assign the number

$$s_i = \frac{1}{2\pi} \oint_{\gamma_i} \alpha \,,$$

- Then $s_1(f_1, \dots, f_n), \dots, s_n(f_1, \dots, f_n)$ are smooth functions on U.
- They coincide (up to a constant) with the action variables.

Example: the *n*-dimensional harmonic oscillator

- Consider the Hamiltonian system $(\mathbb{R}^{2n}, \omega, h)$, where

$$h = \sum_{i=1}^{n} \left(\frac{p_i^2}{2} + \frac{x_i^2}{2} \right), \qquad \omega = \mathrm{d}x_i \wedge \mathrm{d}p_i \,.$$

- It is completely integrable. Indeed, the functions

$$f_i = \frac{p_i^2}{2} + \frac{x_i^2}{2}$$

are integrals, i.e. $\{f_i, h\} = 0$ and $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \neq 0$ a.e.

- The level sets M_Λ are given by

$$M_\Lambda = \big\{ (x_1, \ldots, x_n, p_1, \ldots, p_n) \in \mathbb{R}^{2n} \mid \underbrace{p_i^2 + x_i^2 = 2\Lambda_i}_{\text{circles}} \big\} \simeq \mathbb{T}^n \,.$$

Example: the n-dimensional harmonic oscillator

- We can write

$$h = h(f_1, \ldots, f_n) = \sum_{i=1}^{n} f_i \,.$$

- Let $\varphi^i = \arctan\left(\frac{x_i}{p_i}\right)$. Then,

$$\omega = \mathrm{d}\varphi^i \wedge \mathrm{d}f_i \,.$$

- Thus (φ^i, f_i) are action-angle coordinates.

Example: the n-dimensional harmonic oscillator

- The Hamiltonian vector fields are given by

$$X_{f_i} = \frac{\partial}{\partial \varphi^i}, \quad X_h = \sum_{i=1}^{n} \frac{\partial}{\partial \varphi^i}\,.$$

- Hamilton's equations are given by

$$\begin{aligned} \dot{\varphi}^i &= 1\,, \\ \dot{f}_i &= 0\,. \end{aligned}$$

Hamilton–Jacobi equation from canonical transformations

- Suppose that (q^i, p_i) and (Q^i, P_i) are two sets of Darboux coordinates for $(\mathbb{R}^{2n}, \omega)$, namely,

$$\omega = \underbrace{\mathrm{d}q^i \wedge \mathrm{d}p_i}_{-\mathrm{d}(p_i \mathrm{d}q^i)} = \underbrace{\mathrm{d}Q^i \wedge \mathrm{d}P_i}_{-\mathrm{d}(P_i \mathrm{d}Q^i)}$$

- Then, on an open subset $U \subseteq M$,

$$p_i \mathrm{d}q^i - P_i \mathrm{d}Q^i = \mathrm{d}F\,,$$

for some $F \in C^\infty(U)$.

Hamilton–Jacobi equation from canonical transformations

- Assume that, on some neighbourhood $V \subseteq U$, the Jacobian matrix

$$\frac{\partial(Q, q)}{\partial(p, q)}$$

is not singular.

- Then, we can express $F(q, p) = S(q, Q)$, where S is called the **generating function** of the canonical transformation $(q, p) \mapsto (Q, P)$.
- We have

$$\mathrm{d}S = p_i \mathrm{d}q^i - P_i \mathrm{d}Q^i \rightsquigarrow p_i = \frac{\partial S(q, Q)}{\partial q^i}, \quad P_i = -\frac{\partial S(q, Q)}{\partial Q^i}.$$

Hamilton–Jacobi equation from canonical transformations

- Assume that the canonical transformation is such that the Hamiltonian function in the new coordinates is given by $h = k(P)$, namely,

$$h\left(q^i, \frac{\partial S(q,Q)}{\partial q^i}\right) = k(P_i)\,.$$

- Then, Hamilton's equations are given by

$$\dot{Q}^i = -\frac{\partial k}{\partial P_i}\,, \qquad \dot{P}_i = 0\,.$$

- Since P_i are constants of the motion, by fixing initial conditions we have $h(P) = E = \mathrm{const.}$, obtaining the **Hamilton–Jacobi equation**:

$$h\left(q^i, \frac{\partial S}{\partial q^i}\right) = E\,.$$

Action-angle coordinates from the generating function

- Recall that action variables cab be obtained as

$$s_i = \frac{1}{2\pi} \oint_{\gamma_i} \theta = \frac{1}{2\pi} \oint_{\gamma_i} p_j \mathrm{d} q^j \,,$$

- If the generating function S is **separable**, i.e.

$$S(q^i; Q^i) = S_1(q^1; Q^1, \ldots, Q^n) + \cdots + S_n(q^n; Q^1, \ldots, Q^n) \,,$$

then we can take

$$s_i = s_i(Q^1, \ldots, Q^n) = \frac{1}{2\pi} \oint_{\gamma_i} \frac{\partial S_i}{\partial q^i} \mathrm{d} q^i \,.$$

Action-angle coordinates from the generating function

- We can now regard the action coordinates s_i as “the new Q^i”.
- Let $W = W(q^i, s_i)$ be the generating function of the transformation $(q^i, p_i) \mapsto (s_i, \varphi^i)$.
- Then, angle coordinates are given by

$$\varphi^i = -\frac{\partial W}{\partial s_i}\,.$$

Example: the *n*-dimensional harmonic oscillator

- The Hamilton–Jacobi equation takes the form

$$\frac{1}{2}\sum_{i=1}^{n}\left(\left(\frac{\partial S}{\partial x_i}\right)^2 + x_i^2\right) = E\,.$$

- With the *ansatz* $S = S_1(x_1, E_1) + \cdots + S_n(x_n, E_n)$ and $E = E_1 + \cdots + E_n$, the Hamilton–Jacobi equation is reduced to

$$\left(\frac{\partial S_i}{\partial x_i}\right)^2 + x_i^2 = 2E_i\,.$$

Example: the *n*-dimensional harmonic oscillator

- The solutions of these equations are given by

$$S_i(x_i, E_i) = \pm\frac{1}{2}\left(x_i\sqrt{2E_i - x_i^2} + 2E_i \arctan\left(\frac{x_i}{\sqrt{2E_i - x_i^2}}\right)\right)$$

- Action coordinates are given by

$$s_i = \frac{1}{2\pi}\oint_{\gamma_i}\frac{\partial S_i}{\partial x^i}\mathrm{d}x^i = \frac{1}{2\pi}\int_{-\sqrt{2E_i}}^{\sqrt{2E_i}}\sqrt{2E_i - x_i^2}\,\mathrm{d}x^i = \frac{E_i}{2}$$

Example: the n-dimensional harmonic oscillator

- The new generating function is thus

$$W(x,s)=\sum_{i=1}^{n}\frac{1}{2}\left(x_i\sqrt{4s_i-x_i^2}+4s_i\arctan\left(\frac{x_i}{\sqrt{4s_i-x_i^2}}\right)\right).$$

- Angle coordinates are given by

$$\varphi^i=\frac{\partial S}{\partial s_i}=2\arctan\left(\frac{x_i}{\sqrt{4s_i-x_i^2}}\right)=2\arctan\left(\frac{x_i}{p_i}\right).$$

- Up to multiplicative constant factors, these are the action-angle coordinates we obtained before.

Hamilton–Jacobi equation: the geometric approach

- Consider the Hamiltonian system $(\mathrm{T}^*Q, \omega, h)$
- We want to find a section γ on $\pi_Q : \mathrm{T}^*Q \to Q$ which maps integral curves of $X_h^\gamma := \mathrm{T}\pi_Q \circ X_h \circ \gamma$ into integral curves of X_h, namely,

$$X_h^\gamma \circ \sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(t) \Longrightarrow X_h \circ (\gamma \circ \sigma(t)) = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma \circ \sigma(t)) .$$

- The 1-form γ satisfies this condition iff X_h^γ and X_h are γ-related, i.e.

$$X_h \circ \gamma = \mathrm{T}\gamma \circ X_h^\gamma .$$

- In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{T}^*Q & \xrightarrow{\quad X_h \quad} & \mathrm{TT}^*Q \\ \gamma \uparrow \downarrow \pi_Q & & \mathrm{T}\pi_Q \downarrow \uparrow T\gamma \\ Q & \xrightarrow[\quad X_h^\gamma \quad]{} & \mathrm{T}Q \end{array}$$

Hamilton–Jacobi equation: the geometric approach

- Assuming that $\mathrm{d}\gamma = 0$, the γ-related condition is equivalent to

$$\mathrm{d}(h \circ \gamma) = 0\,. \tag{1}$$

- This can be easily shown by computations in fibred coords. (q^i, p_i).
- Since γ is closed, locally $\gamma = \mathrm{d}S$ and equation (1) can be written as

$$h\left(q^i, \frac{\partial S}{\partial q^i}\right) = E\,,$$

for some (local) constant E.

Hamilton–Jacobi equation: the geometric approach

Theorem (Hamilton–Jacobi theorem)

Let γ be a closed 1-form. Then, the following assertions are equivalent:

1. *If $\sigma : \mathbb{R} \to Q$ is an integral curve of X_h^γ then $\gamma \circ \sigma$ is an integral curve of X_h.*
2. $\mathrm{d}(h \circ \gamma) = 0$,
3. $\operatorname{Im} \gamma$ *is a Lagrangian submanifold of $(\mathrm{T}^* Q, \omega)$ invariant by X_h,*

Hamilton–Jacobi equation: the geometric approach

Proof.

(1) $\Leftrightarrow$ (2) follows from an straightforward computation in fibered coordinates.
(1) $\Leftrightarrow$ (3) Since γ is closed, $\operatorname{Im}\gamma$ is Lagrangian. Moreover,

$$X_h \circ \gamma = \mathrm{T}(\underbrace{\gamma \circ \pi_Q}_{\mathrm{id}_Q}) \circ X_h \circ \gamma = \mathrm{T}\gamma \circ \mathrm{T}\pi_Q \circ X_h \circ \gamma\,,$$

so X_h is tangent to $\operatorname{Im}\gamma$ iff X_h^γ and X_h are γ-related, i.e. iff γ maps every integral curve of $X_h\gamma$ onto an integral curve of X_h. □

Complete solutions

Definition

A **solution of the Hamilton–Jacobi problem** for h is a 1-form $\gamma \in \Omega^1(Q)$ such that

1. $\mathrm{d}\gamma = 0$,
2. $\mathrm{d}(h \circ \gamma) = 0$.

Definition

A **complete solution of the Hamilton–Jacobi problem** for h is a local diffeomorphism $\Phi\colon Q \times \mathbb{R}^n \to \mathrm{T}^*Q$ such that, for each $\lambda \in \mathbb{R}^n$, $\Phi_\lambda = \Phi(\cdot, \lambda)$ is a solution of the Hamilton–Jacobi problem for h.

Complete solutions

- Let Φ be a complete solution of the Hamilton–Jacobi problem for h.
- Let $\pi_i \colon Q \times \mathbb{R}^n \to \mathbb{R}$ denote the projection $\pi_i \colon (q^i, \lambda) \mapsto \lambda_i$.

Proposition

The functions $f_i = \pi_i \circ \Phi^{-1} \colon \mathrm{T}^* Q \to \mathbb{R}$ are constants of the motion in involution.

Complete solutions

Proof.

We know that X_h is tangent to $\operatorname{Im}\Phi_\lambda$ for each $\lambda \in \mathbb{R}^n$, but we can write

$$\operatorname{Im}\Phi_\lambda = \{x \in \mathrm{T}^*Q \mid f_i(x) = \lambda_i\} \ ,$$

and hence $X_h(f_i) = 0$.

Since $\operatorname{Im}\Phi_\lambda$ is Lagrangian and $X_{f_i}(x) \in \mathrm{T}_x(\operatorname{Im}\Phi_\lambda)$,

$$\{f_i, f_j\} = \omega_Q(X_{f_i}, X_{f_j}) = 0 \, .$$

□

Example: the n-dimensional harmonic oscillator

- A complete solution of the Hamilton–Jacobi problem for h is

$$\Phi_E = \mathrm{d}S = \pm \sum_{i=1}^{n} \sqrt{2E_i - x_i^2}\,\mathrm{d}x_i \,.$$

- Its inverse is given by $\Phi^{-1}\colon (x_i, p_i) \mapsto (x_i, E_i(x_i, p_i))$, so the associated constants of the motion are

$$f_i = \pi_i \circ \Phi^{-1} = E_i = \frac{p_i^2}{2} + \frac{x_i^2}{2} \,.$$

- The Lagrangian tori are given by

$$M_\Lambda = \operatorname{Im} \Phi_\Lambda = \{x \in M \mid f_i = \Lambda_i\} \,.$$

KAM theory

- The KAM (Kolmogorov–Arnold–Moser) theorem concerns the stability of completely integrable systems.
- Essentially it says that, under sufficiently small perturbations of the Hamiltonian function of the system, "most" Liouville tori persist.

KAM theory

Definition

An n-tuple $\Omega \in \mathbb{R}^n$ is called

1. **Rationally dependent** if $\Omega \cdot k = 0$ for some $k \in \mathbb{Z}^n$,
2. **Rationally independent** otherwise,
3. **Diophantine** if there exist $L, \gamma > 0$ such that

$$|\Omega \cdot k| \geq \frac{L}{\left(\sum_{i=1}^{n} |k_i|\right)^{\gamma}},$$

for all $k \in \mathbb{Z}^n$.

KAM theory

Theorem (KAM)

Let $H(\varphi, s) = h(s)$ *be an analytic function on* $\mathbb{T}^n \times \mathbb{R}^n$*. Assume that*

1. $\Omega = \frac{\partial h}{\partial s}(s_0)$ *is Diophantine, where* $s_0 \in \mathbb{R}^n$*,*
2. *the Hessian matrix* $\left(\frac{\partial^2 h}{\partial s^i \partial s^j}\right)$ *is non-singular,*
3. P *is an analytic function on* $\mathbb{T}^n \times \mathbb{R}^n$*.*

Then, for sufficiently small $\varepsilon > 0$*, the perturbed system* $H_\varepsilon = H + \varepsilon P$ *admits an invariant torus* $\mathcal{T}$ *close to* $\mathbb{T}^n \times \{s_0\}$ *such that the flow* γ^t *of the perturbed system on* $\mathcal{T}$ *is given by*

$$\psi^{-1} \circ \gamma^t \circ \psi(\varphi_0) = \varphi_0 + \Omega t\,,$$

where $\psi \colon \mathbb{T}^n \to \mathcal{T}$ *is a diffeomorphism.*

Generalizations of Liouville–Arnold theorem

- Liouville–Arnold theorem for non-compact M_Λ (Fiorani *et al.*)
- Liouville–Arnold–Nekhoroshev theorem: partially integrable systems, i.e. with $k < n$ constants of the motion in involution
- Non-abelian integrable systems: $\{f_i, f_j\} \neq 0$
- Singularities: $x \in M$ such that $\operatorname{rank} \mathrm{d}F(x) < n$

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Dziękuję bardzo!

Moltes gràcies!

✉ asier.lopez@icmat.es

🌐 www.alopezgordon.xyz