# Liouville–Arnold theorem for contact Hamiltonian systems

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17th International Young Researchers Workshop
on Geometry, Mechanics and Control
Department of Mathematics, KU Leuven (Belgium)
March, 2023

Financially supported by Grants CEX2019-000904-S and
PID2019-106715GB-C21 funded by MCIN/AEI/10.13039/501100011033

# Outline of the presentation

# Symplectic geometry

- Symplectic geometry is the natural framework for classical mechanics.
- Recall that a symplectic form $\omega$ on $M$ is a 2-form such that $\mathrm{d}\omega = 0$ and $v \mapsto \iota_v \omega$ is an isomorphism.
- Given a function $f$ on $M$, its its Hamiltonian vector field $X_f$ is given by

$$\iota_{X_f}\omega = \mathrm{d}f.$$

- The Poisson bracket $\{\cdot, \cdot\}$ is given by

$$\{f, g\} = \omega(X_f, X_g).$$

## Theorem (Liouville–Arnold theorem)

*Let $f_1, \dots, f_n$ be independent functions in involution (i.e., $\{f_i, f_j\} = 0\ \forall i, j$) on a symplectic manifold $(M^{2n}, \omega)$. Let $M_\Lambda = \{x \in M \mid f_i = \Lambda_i\}$.*

1. *Any compact connected component of $M_\Lambda$ is diffeomorphic to $\mathbb{T}^n$.*
2. *On a neighborhood of $M_\Lambda$ there are coordinates $(\varphi^i, J_i)$ such that*

$$\omega = \mathrm{d}\varphi^i \wedge \mathrm{d}J_i,$$

*and the Hamiltonian dynamics are given by*

$$\frac{\mathrm{d}\varphi^i}{\mathrm{d}t} = \Omega^i(J), \qquad \frac{\mathrm{d}J_i}{\mathrm{d}t} = 0.$$

# Contact geometry

## Definition

A (co-oriented) **contact manifold** is a pair $(M, \eta)$, where $M$ is an $(2n+1)$-dimensional manifold and $\eta$ is a 1-form on $M$ such that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form.

- The contact form $\eta$ defines an isomorphism

$$\begin{aligned} \flat \colon \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto \iota_X \mathrm{d}\eta + \eta(X)\eta, \end{aligned}$$

- There exists a unique vector field $\mathcal{R}$ on $(M, \eta)$, called the **Reeb vector field**, such that $\flat(\mathcal{R}) = \eta$, that is,

$$\iota_{\mathcal{R}} \mathrm{d}\eta = 0, \quad \iota_{\mathcal{R}} \eta = 1.$$

# Contact geometry

- The **Hamiltonian vector field** of $f \in C^\infty(M)$ is given by

$$\flat(X_f) = \mathrm{d}f - (\mathcal{R}(f) + f)\,\eta,$$

- Around each point on $M$ there exist **Darboux coordinates** $(q^i, p_i, z)$ such that

$$\eta = \mathrm{d}z - p_i \mathrm{d}q^i,$$
$$\mathcal{R} = \frac{\partial}{\partial z},$$
$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i \frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z}.$$

# Contact geometry

- The **Jacobi bracket** is given by

$$\{f, g\} = -\mathrm{d}\eta(\flat^{-1}\mathrm{d}f, \flat^{-1}\mathrm{d}g) - f\mathcal{R}(g) + g\mathcal{R}(f).$$

- This bracket is bilinear and satisfies the Jacobi identity.
- However, unlike a Poisson bracket, it does not satisfy the Leibnitz identity:

$$\{f, gh\} \neq \{f, g\}h + \{f, h\}g.$$

# Dissipated quantities

- In contact Hamiltonian dynamics dissipated quantities are akin to conserved quantities in symplectic dynamics.
- Energy (Hamiltonian function) is no longer conserved, but dissipated in a certain manner:

$$X_H(H) = -\mathcal{R}(H)H$$

.

# Dissipated quantities

## Example (linear dissipation)

Let

$$M = \mathbb{R}^3, \quad \eta = \mathrm{d}z - p\mathrm{d}q, \quad H = \frac{p^2}{2} + V(q) + \kappa z.$$

Then $X_H(H) = -\kappa H$, so

$$H(q(t), p(t), z(t)) = e^{-\kappa t} H(q(0), p(0), z(0)).$$

## Definition

An $H$-**dissipated quantity** is a function $f$ on $M$ such that

$$X_H(f) = -\mathcal{R}(H) f.$$

# Dissipated quantities

- A function $f$ is $H$-dissipated iff

$$\{f, H\} = 0.$$

- Noether's theorem: symmetries $\leftrightarrow$ dissipated quantities.

- Let $M_{\langle\Lambda\rangle_+} = \{x \in M \mid \exists\, r \in \mathbb{R}^+ \colon f_\alpha(x) = r\Lambda_\alpha\}$.

## Theorem (Colombo, de León, Lainz, L.-G., 2023)

*Let $(M, \eta)$ be a $(2n+1)$-dimensional contact manifold. Suppose that $f_0, f_1, \ldots, f_n$ are functions in involution such that $(\mathrm{d} f_\alpha)$ has rank at least $n$. Then, $M_{\langle\Lambda\rangle_+}$ is invariant by the Hamiltonian flow of $f_\alpha$ and diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$.*

*Moreover, there is a neighborhood $U$ of $M_{\langle\Lambda\rangle_+}$ such that*

1. *There exists coordinates $(y^0, \ldots, y^n, \tilde{A}_1, \ldots, \tilde{A}_n)$ on $U$ such that the equations of motion are given by*

$$\dot{y}^\alpha = \Omega^\alpha(\tilde{A}_i), \quad \dot{\tilde{A}}_i = 0.$$

2. *There exists a conformal change $\tilde{\eta} = \eta / A_0$ such that $(y^i, \tilde{A}_i, y^0)$ are Darboux coordinates for $(M, \tilde{\eta})$, i.e. $\tilde{\eta} = \mathrm{d} y^0 - \tilde{A}_i \mathrm{d} y^i$.*

# Steps of the proof

1. Symplectize $(M, \eta)$ and $f_\alpha$, obtaining an exact symplectic manifold $(M^\Sigma, \theta)$ and homogeneous functions in involution $f_\alpha^\Sigma$.
2. Prove a Liouville–Arnold theorem for exact symplectic manifolds with homogeneous functions in involution.
3. "Un-symplectize" the action-angle coordinates $(y_\Sigma^\alpha, A_\alpha^\Sigma)$ on $M^\Sigma$, yielding functions $(y^\alpha, A_\Sigma)$ on $M$.
4. Introduce action-angle coordinates $(y^\alpha, \tilde{A}_i)$ on $M$, where $\tilde{A}_i = -\frac{A_i}{A_0}$.

# Exact symplectic manifolds: Liouville geometry

## Definition

An **exact symplectic manifold** is a pair $(M, \theta)$, where $M$ is a manifold and $\theta$ a one-form on $N$ such that $\omega = -\mathrm{d}\theta$ is a symplectic form on $M$.

- The **Liouville vector field** $\Delta$ of $(M, \theta)$ is given by

$$\iota_\Delta \omega = -\theta.$$

- A tensor $T$ is called **homogeneous of degree** $n$ if $\mathcal{L}_\Delta T = nT$.

# Symplectization of contact manifolds

## Definition

Let $(M, \eta)$ be a contact manifold. A **symplectization** is a fibre bundle $\Sigma\colon M^\Sigma \to M$, where $(M^\Sigma, \theta)$ is an exact symplectic manifold, such that

$$\sigma \Sigma^* \eta = \theta,$$

for a function $\sigma$ on $M^\Sigma$ called the **conformal factor**.

# Symplectization of contact manifolds

- Contact geometry $\longleftrightarrow$ Liouville geometry
- Contact form $\eta$ $\longleftrightarrow$ symplectic potential $\theta$
- Functions $\longleftrightarrow$ homogeneous functions of degree 1
- Hamiltonian vector fields $\longleftrightarrow$ Hamiltonian vector fields,
  homogeneous of degree 0

# Symplectization of contact manifolds

**Theorem**

*Given a symplectization* $\Sigma\colon (M^\Sigma, \theta) \to (M, \eta)$ *with conformal factor* $\sigma$*, there is a bijection between functions* $f$ *on* $M$ *and homogeneous functions of degree 1* $f^\Sigma$ *on* $M^\Sigma$ *such that*

$$\Sigma_* (X_{f^\Sigma}) = X_f \,.$$

*This bijection is given by*

$$f^\Sigma = \sigma \Sigma^* f \,.$$

*Moreover, one has*

$$\left\{ f^\Sigma, g^\Sigma \right\}_\theta = \{f, g\}_\eta^\Sigma \,.$$

# Symplectization of contact manifolds

## Example

$\Sigma = \pi_1 \colon (M \times \mathbb{R}^+, \theta = r\eta) \to (M, \eta)$ is a symplectization with conformal factor $\sigma = r$, for $r$ the global coordinate on $\mathbb{R}^+$.

# Liouville–Arnold theorem for exact symplectic manifolds

- We want to obtain action-angle coordinates $(\varphi_\Sigma^\alpha, J_\alpha^\Sigma)$ on $(M^\Sigma, \theta)$ in order to define functions $(\varphi^\alpha, J_\alpha)$ on $(M, \eta)$
- We need homogeneous objects on $(M^\Sigma, \theta)$ so that they have a correspondence with objects on $(M, \eta)$.
- However, the classical Liouville–Arnold theorem does not take into account the homogeneity of $\theta$ and $f_\alpha^\Sigma$.
- Moreover, we need to consider non-compact level sets of $f_\alpha^\Sigma$.

# Liouville–Arnold theorem for exact symplectic manifolds

## Theorem (Colombo, de León, Lainz, L.-G., 2023)

*Let $(M, \theta)$ be an exact symplectic manifold. Suppose that the functions $f_\alpha,\ \alpha = 1, \ldots, n$, on $M$ are independent, in involution and homogeneous of degree 1. Let $U$ be an open neighborhood of $M_\Lambda$ such that:*

1. *$f_\alpha$ have no critical points in $U$,*
2. *the Hamiltonian vector fields of $X_{f_\alpha}$ are complete,*
3. *the submersion $(f_\alpha)\colon U \to \mathbb{R}^n$ is a trivial bundle over $V \subseteq \mathbb{R}^n$.*

*Then, $U \simeq \mathbb{R}^{n-m} \times \mathbb{T}^m \times V$, provided with action-angle coordinates $(y^\alpha, A_\alpha)$ such that*

$$\theta = A_\alpha \mathrm{d}y^\alpha, \qquad \frac{\mathrm{d}y^\alpha}{\mathrm{d}t} = \Omega^\alpha, \qquad \frac{\mathrm{d}A_\alpha}{\mathrm{d}t} = 0.$$

# Sketch of proof

- Since $X_{f_\alpha}$ are $n$ vector fields tangent to $M_\Lambda$, linearly independent and pairwise commutative, they generate the algebra $\mathbb{R}^n$ and $M_\Lambda \simeq \mathbb{R}^n/\mathbb{Z}^k$.
- Thus there are coordinates $y^\alpha = M_\alpha^\beta s^\beta$, where $X_{f_\alpha}(s^\beta) = \delta_\alpha^\beta$.
- The values of $f_\alpha$ define coordinates $(J_\alpha)$ on $V$.
- Since $M_\Lambda$ is Lagrangian, $\theta = A_\alpha(J)\mathrm{d}y^\alpha + B^\alpha(y, J)\mathrm{d}J_\alpha$.
- Since $f_\alpha$ are homogeneous of degree 1, $\theta(X_{f_\alpha}) = f_\alpha$.
- By construction, $\Delta(y^\alpha) = 0$.
- With additional contractions with $\theta$ and $\omega$, one concludes that $\theta = A_\alpha \mathrm{d}y^\alpha$, where $J_\beta = M_\beta^\alpha J_\alpha$.

# From conditions on $f_\alpha^\Sigma$ to conditions on $f_\alpha$

- In order to apply the Liouville–Arnold theorem for exact symplectic manifolds, we need to translate the conditions on $f_\alpha$ to conditions on $f_\alpha^\Sigma$.
- Let $M_{\langle\Lambda\rangle_+} = \{x \in M \mid \exists\, r \in \mathbb{R}^+ \colon F(x) = r\Lambda\}$.
- Let $F^\Sigma = (f_\alpha^\Sigma)$ and $\tilde{M}_\Lambda = (F^\Sigma)^{-1}(\Lambda)$.
- Given the functions $f_0, f_1, \ldots, f_n \colon M \to \mathbb{R}$, let $F = (f_\alpha)$ and

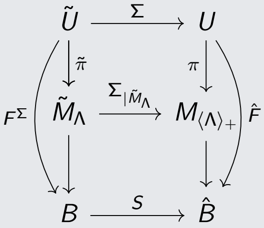
$$\hat{F} = S \circ F \colon M \to \mathbb{S}^n,$$

where $S \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{S}^n$ denotes the projection on the sphere.

# From conditions on $f_\alpha^\Sigma$ to conditions on $f_\alpha$

## Lemma

*Given $\langle\Lambda\rangle_+ \in S^n$, let $\hat{B} \subseteq S^n$ be an open neighborhood of $\langle\Lambda\rangle_+$ and let $\pi\colon U \to M_{\langle\Lambda\rangle_+}$ be a tubular neighborhood of $M_{\langle\Lambda\rangle_+}$ such that $\hat{F}_{|U}\colon U \to \hat{B}$ is a submersion with diffeomorphic fibers. Define $B = S^{-1}(\hat{B})$ and $\tilde{U} = \Sigma^{-1}(U)$ and $\tilde{\pi} = \Sigma^{-1}_{\tilde{M}_\Lambda} \circ \pi \circ \Sigma$. Then, $\tilde{\pi}\colon \tilde{U} \to \tilde{M}_\Lambda$ is a tubular neighborhood of $\tilde{M}_\Lambda$ such that $F^\Sigma_{|\tilde{U}}\colon \tilde{U} \to B$ is a submersion with diffeomorphic fibers.*



## Theorem (Colombo, de León, Lainz, L.-G., 2023)

*Let $(M, \eta)$ be a $(2n+1)$-dimensional contact manifold. Suppose that $f_0, f_1, \ldots, f_n$ are functions in involution such that $(\mathrm{d}f_\alpha)$ has rank at least $n$. Assume that the Hamiltonian vector fields $X_{f_\alpha}$ are complete. Given $\Lambda \in \mathbb{R}^{n+1} \setminus \{0\}$, let $\hat{B} \subseteq S^n$ be an open neighborhood of $\langle \Lambda \rangle_+$ and let $\pi \colon U \to M_{\langle \Lambda \rangle_+}$ be a tubular neighborhood of $M_{\langle \Lambda \rangle_+}$ such that $\hat{F}_{|U} \colon U \to \hat{B}$ is a submersion with diffeomorphic fibers. Then*

1. *$M_{\langle \Lambda \rangle_+}$ is invariant by the Hamiltonian flow of $f_\alpha$ and diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$.*
2. *There exists coordinates $(y^0, \ldots, y^n, \tilde{A}_1, \ldots, \tilde{A}_n)$ on $U$ such that the equations of motion are given by*

$$\dot{y}^\alpha = \Omega^\alpha, \quad \dot{\tilde{A}}_i = 0.$$

3. *There exists a conformal change $\tilde{\eta} = \eta / A_0$ such that $(y^i, \tilde{A}_i, y^0)$ are Darboux coordinates for $(M, \tilde{\eta})$.*

# Sketch of the proof

1. Symplectize $(M, \eta)$ and $f_\alpha$, in order to apply the Liouville–Arnold theorem for exact symplectic manifolds
   - $\{f_\alpha, f_\beta\} = 0 \Rightarrow \{f_\alpha^\Sigma, f_\beta^\Sigma\} = 0$.
   - $X_{f_\alpha}$ complete $\Rightarrow X_{f_\alpha^\Sigma}$ complete.
   - $\operatorname{rank} \mathrm{d} f_\alpha \geq n \Rightarrow \operatorname{rank} \mathrm{d}(\underbrace{\sigma \Sigma^* f_\alpha}_{f_\alpha^\Sigma}) \geq n+1$.
   - $\Sigma\left((F^\Sigma)^{-1}(\Lambda)\right) = \left\{x \in M \mid \exists s \in \mathbb{R}^+ : \ F(x) = \frac{\Lambda}{s}\right\} = M_{\langle\Lambda\rangle_+}$.
   - $X_{f_\alpha}$ commute and are tangent to $M_{\langle\Lambda\rangle_+} \Rightarrow M_{\langle\Lambda\rangle_+} \simeq \mathbb{T}^k \times \mathbb{R}^{n+1-k}$.
2. "Un-symplectize" the action-angle coordinates $(y_\Sigma^\alpha, A_\alpha^\Sigma)$ on $\tilde{U}$, yielding functions $(y^\alpha, A_\alpha)$ on $U$.
3. Introduce action-angle coordinates $(y^\alpha, \tilde{A}_i)$ on $U$
   - Since $\Lambda \neq 0$, $\exists A_\alpha \neq 0$. W.l.o.g., assume $A_0 \neq 0$.
   - Then $\left(y^\alpha, \tilde{A}_i = -\frac{A_i}{A_0}\right)$ are coordinates on $U$.

# Sketch of the proof

- By construction, $y^\alpha$ are linear combinations of flows of $X_{f_\alpha}$, namely,

$$X_{f_\alpha} = M^\alpha_\beta \frac{\partial}{\partial s^\beta}.$$

- Therefore, the dynamics are given by

$$\frac{\mathrm{d}y^\alpha}{\mathrm{d}t} = \Omega^\alpha, \qquad \frac{\mathrm{d}\tilde{A}_i}{\mathrm{d}t} = 0.$$

- $\theta^\Sigma = A^\Sigma_\alpha \mathrm{d}y^\alpha_\Sigma \rightsquigarrow \eta = A_\alpha \mathrm{d}y^\alpha$, so

$$\tilde{\eta} = \frac{1}{A_0}\eta = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i.$$

# Other notions of integrability

- Khesin and Tabachnikov, Liberman, Banyaga and Molino, Lerman, etc. have defined notions of contact complete integrability which are geometric but not dynamical, e.g. a certain foliation over a contact manifold.
- Boyer considers the so-called good Hamiltonians $H$, i.e., $\mathcal{R}(H) = 0 \rightsquigarrow$ no dissipated quantities, "symplectic" dynamics.
- Miranda considered integrability of the Reeb dynamics when $\mathcal{R}$ is the generator of an $S^1$-action.
- We are interested in complete integrability of contact Hamiltonian dynamics.

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# Thanks for your attention!

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