On the stability of contact Hamiltonian systems

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Outline of the presentation

1. Introduction
2. Contact Hamiltonian systems
3. Stability
4. Inverse problem

Stability of Hamiltonian systems

- Symplectic geometry is the natural framework for classical mechanics.
- Recall that symplectic manifold is a pair (M, ω), where M is a manifold and ω is a 2-form on M such that $\mathrm{d}\omega = 0$ and $\mathrm{T}M \ni v \mapsto \iota_v \omega \in \mathrm{T}^* M$ is an isomorphism.
- Given a function h on M, its Hamiltonian vector field X_h is given by

$$\iota_{X_h} \omega = \mathrm{d}h \,.$$

Stability of Hamiltonian systems

- In some cases, the stability properties of X_h can be studied by analyzing the maxima and minima of h.
- This method is based on the fact that h is a conserved quantity w.r.t. X_h, namely, $X_h(h) = 0$.
- As we will see, analogue techniques do not make sense for contact Hamiltonian systems, since their Hamiltonian functions are not, in general, conserved quantities.

Why contact Hamiltonian systems?

- While (symplectic) Hamiltonian dynamics is conservative, contact Hamiltonian dynamics permits modeling certain dissipative systems.
- Certain dynamical systems on an odd-dimensional manifolds can be regarded as a contact Hamiltonian vector field.
- Around a point where $x \in M$ such that $X(x) \neq 0$, every vector field X on an even-dimensional manifold is locally a Hamiltonian vector field with respect to some symplectic form. However, this is not necessarily the case if $X(x) = 0$.
- For instance, $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is not Hamiltonian w.r.t. any symplectic form on $\mathbb{R}^2$, but it is a contact Hamiltonian vector field w.r.t. a contact form in $\mathbb{R}^3$.

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Contact geometry

Definition

A (co-oriented) **contact manifold** is a pair (M, η), where M is an $(2n+1)$-dimensional manifold and η is a 1-form on M such that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form.

- The contact form η defines an isomorphism

$$\begin{aligned} \flat \colon \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto \iota_X \mathrm{d}\eta + \eta(X)\eta, \end{aligned}$$

- There exists a unique vector field $\mathcal{R}$ on (M, η), called the **Reeb vector field**, such that $\flat(\mathcal{R}) = \eta$, that is,

$$\iota_{\mathcal{R}} \mathrm{d}\eta = 0, \quad \iota_{\mathcal{R}} \eta = 1.$$

Contact geometry

- The **Hamiltonian vector field** of $f \in C^\infty(M)$ is given by

$$\flat(X_f) = \mathrm{d}f - (\mathcal{R}(f) + f)\,\eta,$$

- Around each point on M there exist **Darboux coordinates** (q^i, p_i, z) such that

$$\eta = \mathrm{d}z - p_i \mathrm{d}q^i,$$
$$\mathcal{R} = \frac{\partial}{\partial z},$$
$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i \frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z}.$$

Contact Hamiltonian systems

Definition

A **contact Hamiltonian system** is a triple (M, η, h) formed by a contact manifold (M, η) and a **Hamiltonian function** $h \in C^\infty(M)$.

- The dynamics of (M, η, h) is determined by the integral curves of the Hamiltonian vector field X_h of h w.r.t. η.

Contact Hamiltonian systems

- In Darboux coordinates, these curves $c(t) = (q^i(t), p_i(t), z(t))$ are determined by the **contact Hamilton equations**:

$$
\begin{aligned}
\frac{\mathrm{d}q^i(t)}{\mathrm{d}t} &= \frac{\partial h}{\partial p_i} \circ c(t)\,, \\
\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= -\frac{\partial h}{\partial q^i} \circ c(t) + p_i(t)\frac{\partial h}{\partial z} \circ c(t)\,, \\
\frac{\mathrm{d}z(t)}{\mathrm{d}t} &= p_i(t)\frac{\partial h}{\partial p_i} \circ c(t) - h \circ c(t)\,.
\end{aligned}
$$

Dissipated quantities

- In contact Hamiltonian dynamics dissipated quantities are akin to conserved quantities in symplectic dynamics.
- Energy (Hamiltonian function) is no longer conserved, but dissipated in a certain manner:

$$X_h(h) = -\mathcal{R}(h)h\,.$$

Dissipated quantities

Example (linear dissipation)

Let

$$M = \mathbb{R}^3, \quad \eta = \mathrm{d}z - p\mathrm{d}q, \quad h = \frac{p^2}{2} + V(q) + \kappa z.$$

Then $X_h(h) = -\kappa h$, so

$$h \circ c(t) = e^{-\kappa t} h \circ c(0)\,,$$

along an integral curve c of X_h.

Dissipated quantities

Definition

Let (M, η, h) be a contact Hamiltonian system. A **dissipated quantity** is a solution $f \in C^\infty(M)$ to the PDE

$$X_h(f) = -\mathcal{R}(h)f \, .$$

- Noether's theorem: symmetries $\leftrightarrow$ dissipated quantities.

Equilibrium points

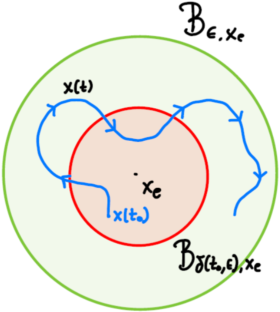
- Let M be an n-dimensional manifold
- The solutions of the system of ODEs

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = X^i(x)\,, \qquad i = 1, \ldots, n\,,$$

are the integral curves of the vector field $X = X^i \frac{\partial}{\partial x^i}$.

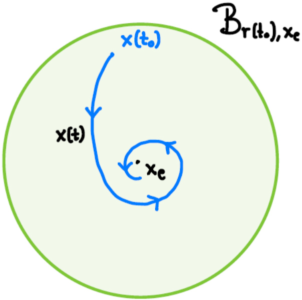
- An **equilibrium point** is a point $x_e \in M$ such that $X(x_e) = 0$.

Stable equilibrium points



If $M = \mathbb{R}^n$, an equilibrium point x_e of X is called **stable** if, for every $t_0 \in \mathbb{R}$ and any ball B_{ϵ, x_e}, there exists a ball $B_{\delta(\epsilon), x_e}$, such that every integral curve $x(t)$ of X with $x(t_0) \in B_{\delta(\epsilon), x_e}$ satisfies that $x(t) \in B_{\epsilon, x_e}$ for all times $t \geq t_0$.

Asymptotically stable equilibrium points



An equilibrium point $x_e \in \mathbb{R}^n$ is **asymptotically stable** if x_e is stable and there exists an open neighbourhood B_{r,x_e} of x_e such that every integral curve $x(t)$ of X with some t_0 satisfying $x(t_0) \in B_{r,x_e}$ converges to x_e.

How to extend this to manifolds?

- The existence of partitions of unity implies that every differentiable manifold can be endowed with a Riemannian metric induced by the Euclidean metric.
- Moreover, the topology induced by the Riemannian metric coincides with the topology of the manifold.
- This implies that a coordinate neighbourhood U is homeomorphic to an open subset in $\mathbb{R}^n$ with the Euclidean norm.
- We will identify balls in $\mathbb{R}^n$ with the neighbourhoods in U to which they are homeomorphic.

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Lyapunov functions

Theorem

Let $X \in \mathfrak{X}(M)$ *be a vector field such that* $X(x_0) = 0$. *If there exists a function* $V : U \to \mathbb{R}$, *defined on some open neighbourhood* U *of* x_0 *such that*

1. $V(x_0) = 0$ *and* $V(x) > 0$ *for* $x \in U \setminus \{x_0\}$,
2. $\dot{V}(x) = (XV)(x) \leq 0$ *for* $x \in U \setminus \{x_0\}$,

then x_0 *is stable. If additionally* $\dot{V}(x) < 0$ *for* $x \in U \setminus \{x_0\}$, *then* x_0 *is asymptotically stable.*

Lyapunov functions

Definition

A function $V\colon U \to \mathbb{R}$ satisfying ① and ② is called a **Lyapunov function**. If $\dot{V}(x) < 0$ for $x \in U \setminus \{x_0\}$, the function V is called a **strict Lyapunov function**.

Dissipated quantities and stability

Proposition (de Lucas, L.-G., Zawora)

Let (M, η, h) be a contact Hamiltonian system such that $X_h(x_0) = 0$. Suppose that $f_1, \ldots, f_k$ are dissipated quantities. If $(\mathcal{R}h)(x_0) > 0$ at an isolated point $x_0 \in \bigcap_{i=1}^k f_i^{-1}(0)$, then x_0 is asymptotically stable.

Proof.

There exists a neighbourhood U of x_0 where $\mathcal{R}h > 0$ and such that $\bigcap_{i=1}^k f_i^{-1}(0) \cap U = \{x_0\}$. By construction,

$$V : x \in U \mapsto \sum_{i=1}^k f_i^2(x) \in \mathbb{R}$$

is a strict Lyapunov function. □

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Example

- Consider the contact Hamiltonian system $(\mathbb{R}^3, \eta, h)$, with

$$\eta = \mathrm{d}z - p\mathrm{d}q\,, \quad h = \frac{p^2}{2} + \frac{q^2}{2} + z\,.$$

- The Hamiltonian vector field of h is

$$X_h = p\frac{\partial}{\partial q} - (q+p)\frac{\partial}{\partial p} + \left(\frac{p^2}{2} - \frac{q^2}{2} - z\right)\frac{\partial}{\partial z}\,,$$

  which vanishes at 0.
- The function $f = z - \frac{pq}{2}$ is a dissipated quantity.
- We have that $h^{-1}(0) \cap f^{-1}(0) = \{0\}$.
- Since $\mathcal{R}h = 1$ everywhere (in particular, $\mathcal{R}h(0) > 0$), it follows that 0 is an asymptotically stable equilibrium point of X_h.

Necessary condition for being an isolated point

Proposition

Let $f_1, \ldots, f_k \in C^\infty(M)$ *be such that* $f_i(x_0) = 0$ *for* $i = 1, \ldots, k$ *and* $\dim M \geq k + 1$. *If* x_0 *is an isolated point of* $\bigcap_{i=1}^k f_i^{-1}(0)$, *then*

$$\mathrm{d}f_1\big|_{x_0} \wedge \cdots \wedge \mathrm{d}f_k\big|_{x_0} = 0.$$

Necessary condition for being an isolated point

Proof.

Suppose that $\mathrm{d}f_1|_{x_0} \wedge \cdots \wedge \mathrm{d}f_k|_{x_0} \neq 0$. Then, on some neighbourhood U of x_0, the map $\Phi\colon U \ni x \mapsto (f_1(x), \ldots, f_k(x)) \in \mathbb{R}^k$ is regular, and hence $\mathrm{d}f_1|_U \wedge \cdots \wedge \mathrm{d}f_k|_U \neq 0$. Thus,

$$\Phi^{-1}(0) = f_1^{-1}(0) \cap f_2^{-1}(0) \cap \cdots \cap f_k^{-1}(0) \cap U$$

is a k-codimensional submanifold and x_0 is not an isolated point of $\bigcap_{i=1}^k f_i^{-1}(0)$. □

Sufficient condition for being an isolated point

Proposition

Let $f_1, \dots, f_k \in C^\infty(M)$ *with* $k < n$ *be such that* $f_i(x_0) = 0 \ \forall\, i = 1, \dots, k$*, and* $\dim\langle \mathrm{d} f_i|_{x_0} \rangle = k - 1$*. W.l.o.g., assume that* $\mathrm{d} f_1|_{x_0}, \dots, \mathrm{d} f_{k-1}|_{x_0}$ *are linearly independent. If* $g = f_k + \lambda_1 f_1 + \cdots + \lambda_{k-1} f_{k-1}$*, where* $\lambda_1, \dots, \lambda_{k-1}$ *are Lagrange multipliers, has a strict minimum or maximum at* x_0*, then* x_0 *is an isolated point of* $\bigcap_{i=1}^k f_i^{-1}(0)$.

Sufficient condition for being an isolated point

Proof.

By construction, $g(x_0) = 0$. If x_0 is a constrained local strict minimum or maximum of g, then there exists a neighbourhood U of x_0 in $f_1^{-1}(0) \cap \ldots \cap f_{k-1}^{-1}(0)$ such that $g(x) \neq 0$ for all $x \in U \setminus \{x_0\}$. Consequently, $f_1(x), \ldots, f_k(x)$ cannot vanish simultaneously at any $x \in U \setminus \{x_0\}$. We conclude that

$$\bigcap_{i=1}^{k} f_i^{-1}(0) \cap \widehat{U} = \{x_0\}$$

for any open subset $\widehat{U}$ in M such that $\widehat{U} \cap \bigcap_{i=1}^{k-1} f_i^{-1}(0) = U$. □

Theorem (de Lucas, L.-G., Zawora)

Let (M, η, h) *be a contact Hamiltonian system and let* x_0 *be an equilibrium point of* X_h*. Suppose that* $f_1, \ldots, f_k \in C^\infty(M)$ *are dissipated quantities for* X_h *such that* $f_i(x_0) = 0$ *for* $i = 1, \ldots, k$*, and* $\dim\langle \mathrm{d}f_1|_{x_0}, \ldots, \mathrm{d}f_k|_{x_0}\rangle = k - 1$*. W.l.o.g., assume that* $\mathrm{d}f_1|_{x_0}, \ldots, \mathrm{d}f_{k-1}|_{x_0}$ *are linearly independent. If the function* $g = f_k + \lambda_1 f_1 + \cdots + \lambda_{k-1} f_{k-1}$*, where* $\lambda_1, \ldots, \lambda_{k-1}$ *are Lagrange multipliers, has a strict minimum or maximum at* x_0*, then* x_0 *is asymptotically stable.*

Computation of the Lagrange multipliers

Since $\mathrm{d}f_1|_{x_0}, \ldots, \mathrm{d}f_{k-1}|_{x_0}$ are indep., and $\dim\langle \mathrm{d}f_1|_{x_0}, \ldots, \mathrm{d}f_k|_{x_0}\rangle = k-1$, one has that

$$\mathrm{d}f_k|_{x_0} = -\sum_{i=1}^{k-1} \lambda_i \mathrm{d}f_i|_{x_0} .$$

Conditions for being an equilibrium point

Proposition

Let (M, η, h) be a contact Hamiltonian system. Then, $x_0 \in M$ is an equilibrium point of (M, η, h) if and only if $h(x_0) = 0$ and $\mathrm{d}h|_{\ker \eta_{x_0}} = 0$.

Combining the previous Theorem and Proposition, we have the following.

Proposition (de Lucas, L.-G., Zawora)

Let (M, η, h) *be a contact Hamiltonian system and* $f \in C^\infty(M)$ *a dissipated quantity. Consider an equilibrium point* $x_0 \in M$ *such that*

1. $f(x_0) = 0,\ \mathrm{d}f|_{x_0} \neq 0$
2. $\mathrm{d}f|_{x_0} + \lambda \mathrm{d}h|_{x_0} = 0,$
3. x_0 *is a strict maximum or minimum of the function* $\tilde{f} = f + \lambda h$*.*

Then, x_0 *is an asymptotically stable equilibrium point of* X_h*.*

The inverse problem for contact Hamiltonian systems

Given a vector field X on a $(2n+1)$-dimensional manifold M, does it exist a (local) contact form η on M such that X is (locally) a Hamiltonian vector field w.r.t. η?

The inverse problem outside of zeroes

- In a neighbourhood U of a point $x \in M$ where $X(x) \neq 0$ this is always the case.
- Indeed, we can choose coordinates $(x^1, \ldots, x^n, y_1, \ldots, y_n, z)$ in U such that $X = \frac{\partial}{\partial z}$.
- $\eta = \mathrm{d}z - y_i \mathrm{d}x^i \in \Omega^1(U)$ is a contact form on U.
- Moreover, $X = \mathcal{R}$ is the Reeb vector field of η
- In other words, $X = X_h$ is the Hamiltonian vector field of $h \equiv -1$.

Example

- The SIR epidemiological model can be described by the vector field

$$X = -\frac{\beta IS}{N}\frac{\partial}{\partial S} + \left(\frac{\beta IS}{N} - \gamma I\right)\frac{\partial}{\partial I} + \gamma I \frac{\partial}{\partial R} \in \mathfrak{X}(\mathbb{R}^3_+)\,,$$

  where S, I and R denote the susceptible, infected and recovered (or deceased) population, respectively.
- It is well-known that the total population $N := S + I + R$ is conserved, i.e., $X(N) = 0$.
- The function $g = S + I - \frac{\gamma N \log S}{\beta}$ is also conserved.
- $X = \frac{\partial}{\partial \widehat{S}}$ is the Reeb vector field of $\eta = \mathrm{d}\widehat{S} - N\mathrm{d}g$, where

$$\widehat{S} = -\int \frac{N}{\beta S\left(g - S + \frac{\gamma N \log S}{\beta}\right)}\,\mathrm{d}S = -\int \frac{N}{\beta SI}\,\mathrm{d}S\,.$$

The inverse problem around an equilibrium point

- In a neighbourhood U of $x_0 \in M$ such that $X(x_0) = 0$ the problem becomes highly non-trivial, since we can lo longer make of the straightening theorem.
- More precisely, our open problem is as follows.

The inverse problem around an equilibrium point

Problem

Consider a linear vector field on $\mathbb{R}^3$*, namely,*

$$X = \sum_{i=1}^{3} \sum_{j=1}^{3} k_{ij} x^i \frac{\partial}{\partial x^j} ,$$

where k_{ij} *are constants and* (x^i) *are the canonical coordinates. Does it exist a contact form* $\eta \in \Omega^1(U)$ *on a neighbourhood* $U \ni 0$ *such that* X *is a Hamiltonian vector field w.r.t.* η*?*

So far we only know the answer in some particular cases, such as the following:

- If $X = kx\frac{\partial}{\partial x}$, $\nexists$ contact form making X a contact Hamiltonian vector field.
- If $X = k_1 x\frac{\partial}{\partial x} + k_2 y\frac{\partial}{\partial y} + k_3\frac{\partial}{\partial z}$ and $k_1, k_2, k_3 > 0$ (or < 0), $\nexists$ contact form making X a contact Hamiltonian vector field.
- $X = X_h = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is the contact Hamiltonian vector field of $h = -x - zy$ w.r.t. $\eta = dx + zdy + yzdz$.

¡Gracias por vuestra atención!

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