Reduction of hybrid forced mechanical systems

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Introduction

- Hybrid systems are dynamical systems with continuous-time and discrete-time components on its dynamics.
- They can be used to model mechanical systems with collisions.
- In the control community, simple hybrid systems have been mainly employed for the understanding of locomotion gaits in bipeds and insects.
- In our case, the continuous dynamics will be the one corresponding to a forced mechanical system.
- External forces appear in many dynamical systems:
    - systems with dissipation or friction,
    - control forces,
    - nonholonomic Čaplygin systems.

Forced Hamilton equations

- As it is well-known, T^*Q is endowed with a canonical symplectic form $\omega_Q = -\mathrm{d}\theta_Q$, where $\theta_Q = p_i \mathrm{d}q^i$ in Darboux coordinates.
- A **forced Hamiltonian system** is a triple (Q, H, α), where
  - Q is an n-dimensional differentiable manifold,
  - $H\colon \mathrm{T}^*Q \to \mathbb{R}$ is a Hamiltonian function, and
  - α is a semibasic 1-form on T^*Q (i.e., $\alpha(X) = 0 \ \forall\ X \in V(\mathrm{T}^*Q)$).
- The **forced Hamiltonian vector field** $X_{H,\alpha}$ is given by

$$\iota_{X_{H,\alpha}} \omega_Q = \mathrm{d}H + \alpha.$$

- Its integral curves satisfy the **forced Hamilton equations**:

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i},$$
$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i} - \alpha_i.$$

Simple hybrid systems

- A **simple hybrid system** is a tuple $\mathscr{H} = (D, X, S, \Delta)$, where:
  - D is a smooth manifold, called the **domain**, where the dynamics evolve,
  - X is a smooth vector field on D, representing the dynamics,
  - S is an embedded submanifold of D with co-dimension 1, called the **switching surface**, which can be thought of as the wall where the impact takes place,
  - $\Delta : S \to D$ is a smooth embedding, called the **impact map**, which represents the dynamics at the instant of the impact.
- Its dynamics are given by:

$$\begin{cases} \dot{\gamma}(t) = X(\gamma(t)), & \gamma(t) \notin S, \\ \gamma^+(t) = \Delta(\gamma^-(t)) & \gamma^-(t) \in S, \end{cases}$$

where $\gamma^{\pm}(t) := \lim_{\tau \to t^{\pm}} x(\tau)$.

Simple hybrid forced Hamiltonian systems

- A simple hybrid system is a **simple hybrid forced Hamiltonian system** if it is of the form $\mathscr{H} = (\mathrm{T}^*Q, X_{H,\alpha}, S, \Delta)$, where $X_{H,\alpha}$ is the dynamical vector field of the forced Hamiltonian system (Q, H, F).
- Suppose that $H(q,p) = ||p||_q^2 + V(q)$, where $||\cdot||_q$ denotes the norm at T_q^*Q defined by a pseudo-Riemannian metric on Q.
- Given a smooth constraint function $h : Q \to \mathbb{R}$ with regular value 0, we can construct a domain, a switching surface and an impact map explicitly explicitly:
  - $D = \{(q,p) \in \mathrm{T}^*Q : h(q) \geq 0\}$,
  - $S = \{(q,p) \in \mathrm{T}^*Q : h(q) = 0,\ \langle\langle p, \mathrm{d}h_q\rangle\rangle_q \geq 0\}$,
  - $\Delta(q,p) = (q, P(q,p))$, where

$$P(q,p) = p - (1+e)\frac{\langle\langle p, \mathrm{d}h_q\rangle\rangle_q}{||\mathrm{d}h_q||_q^2}\mathrm{d}h_q,$$

and e is the elastic constant.

Cotangent action and momentum map

- Consider a free and proper Lie group action $\Phi\colon G \times Q \to Q$.
- Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ its dual.
- The cotangent lift of this action is given by $\Phi^{\mathrm{T}^*Q}\colon (g,(q,p)) \mapsto (\mathrm{T}^*\Phi_{g^{-1}}(q,p))$.
- Φ^{T^*Q} admits an Ad^*-equivariant momentum map

$$J\colon \mathrm{T}^*Q \to \mathfrak{g}^*$$
$$J(\alpha_q)(\xi) = \alpha_q(\xi_Q(q))$$

Here ξ_Q is the infinitesimal generator of the action of $\xi \in \mathfrak{g}$ on Q:

$$\xi_Q = \left.\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\exp t\xi}\right|_{t=0}$$

- Consider a forced simple hybrid forced Hamiltonian system $\mathscr{H} = (\mathrm{T}^*Q, X_{H,\alpha}, S_H, \Delta_H)$ and a Lie group action $\Phi\colon G \times Q \to Q$ such that
  - H is invariant under Φ^{T^*Q}, i.e. $H \circ \Phi^{\mathrm{T}^*Q} = H$,
  - Φ^{T^*Q} restricts to an action of G on S_H,
  - Δ_H is equivariant with respect to the previous action, namely

$$\Delta_H \circ \Phi_g^{\mathrm{T}^*Q} |_{S_H} = \Phi_g^{\mathrm{T}^*Q} \circ \Delta_H.$$

- A momentum map J will be called a **generalized hybrid momentum map** for $\mathscr{H}$ if, for each regular value μ_- of J,

$$\Delta_H \left(J|_{S_H}^{-1}(\mu_-) \right) \subset J^{-1}(\mu_+),$$

  for some regular value μ_+.
- In other words, for every point in the switching surface such that the momentum before the impact takes a value of μ_-, the momentum will take a value μ_+ after the impact.

- Consider a regular value μ_i of J. Let G_{μ_i} denote the isotropy subgroup in μ_i under the co-adjoint action.

Lemma

If Δ_H is equivariant with respect to Φ^{T^*Q}, and $\mu_1,\ \mu_2$ are regular values of J such that $\Delta_H\left(J|_{S_H}^{-1}(\mu_1)\right) \subset J^{-1}(\mu_2)$, then $G_{\mu_1} = G_{\mu_2}$.

- Suppose that H and α are G-invariant and assume that J is a generalized hybrid momentum map.
- Consider a sequence $\{\mu_i\}$ of regular values of J, such that $\Delta_H\left(J|_{S_H}^{-1}(\mu_i)\right) \subset J^{-1}(\mu_{i+1})$.

Theorem

i. *Each reduced space is a symplectic manifold* $(M_{\mu_i}, \omega_{\mu_i})$, *where* $M_{\mu_i} := J_\alpha^{-1}(\mu_i)/G_{\mu_0}$ *and* $\pi_{\mu_i}^* \omega_{\mu_i} = i_{\mu_i}^* \omega_Q$.
ii. H *and* α *induce a reduced forced Hamiltonian system* $(H_{\mu_i}, \alpha_{\mu_i})$ *on* M_{μ_i}, *given by* $H_{\mu_i} \circ \pi_{\mu_i} = H \circ i_{\mu_i}$ *and* $\pi_{\mu_i}^* \alpha_{\mu_i} = i_{\mu_i}^* \alpha$.
iii. $J_\alpha |_{S_H}^{-1} (\mu_i) \subset S_H$ *reduces to a submanifold of the reduced space* $(S_H)_{\mu_i} \subset J_\alpha^{-1}(\mu_i)/G_{\mu_0}$.
iv. $\Delta_{H|J^{-1}(\mu_i)}$ *reduces to a map* $(\Delta_H)_{\mu_i} : (S_H)_{\mu_i} \to J_\alpha^{-1}(\mu_{i+1})/G_{\mu_0}$.

Therefore, after the reduction procedure, we get a sequence of reduced simple hybrid forced Hamiltonian systems $\{\mathscr{H}_\alpha^{\mu_i}\}$, where $\mathscr{H}_\alpha^{\mu_i} = (J_\alpha^{-1}(\mu_i)/G_{\mu_0}, X_{H_{\mu_i},\alpha_{\mu_i}}, (S_H)_{\mu_i}, (\Delta_H)_{\mu_i})$.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & J_\alpha^{-1}(\mu_i) & \hookleftarrow & J_\alpha|_{S_H}^{-1}(\mu_i) & \xrightarrow{\Delta_{H|J^{-1}(\mu_i)}} & J_\alpha^{-1}(\mu_{i+1}) & \hookleftarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \dfrac{J_\alpha^{-1}(\mu_i)}{G_{\mu_0}} & \hookleftarrow & (S_H)_{\mu_i} & \xrightarrow{(\Delta_H)_{\mu_i}} & \dfrac{J_\alpha^{-1}(\mu_{i+1})}{G_{\mu_0}} & \hookleftarrow \cdots
\end{array}$$

Rolling disk with dissipation hitting fixed walls I

- Consider a homogeneous circular disk of radius R and unit mass moving in the vertical plane xOy. Let (x, y) be the coordinates of the centre of the disk and φ the angle between a point of the disk and the axis Oy.
- The dynamics of the disk is determined by the forced Hamiltonian system $(\mathbb{R}^2 \times \mathbb{S}^1, H, \alpha)$, where

$$
\begin{aligned}
H &= \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2k^2} p_\varphi^2, \\
\alpha &= -2c(p_x xy - p_y x^2)\mathrm{d}x + 2c(p_y xy - p_x y^2)\mathrm{d}y,
\end{aligned}
$$

subject to the nonholonomic constraint of rolling without sliding, namely, $p_x = \dfrac{R}{k^2} p_\varphi$.

Rolling disk with dissipation hitting fixed walls II

- Consider the Lie group action of $\mathbb{S}^1 \times \mathbb{S}^1$ on Q given by $(x, y, \varphi) \mapsto (\cos\psi_1\ x - \sin\psi_1\ y, \sin\psi_1\ x + \cos\psi_1\ y, \varphi + \psi_2)$.
- Note that H and α are invariant under the lifted action on $\mathrm{T}^*(\mathbb{R}^2 \times \mathbb{S}^1)$. The corresponding momentum map is

$$J(x, y, \varphi, p_x, p_y, p_\varphi) = (xp_y - yp_x, p_\varphi).$$

- Suppose that there are two rough walls at the axis $y = 0$ and at $y = h$, where $h = \alpha R$ for some constant $\alpha > 1$.
- When the disk hits one of the walls, the impact map is given by

$$\Delta\colon \left(p_x^-, p_y^-, p_\varphi^-\right) \mapsto \left(\frac{R^2 p_x^- + k^2 R p_\varphi^-}{k^2 + R^2}, -p_y^-, \frac{R p_x^- + k^2 p_\varphi^-}{k^2 + R^2}\right).$$

Rolling disk with dissipation hitting fixed walls III

- One can check that J is a generalized hybrid momentum map but not a hybrid momentum map, i.e., $J(q_1, p_1^-) = J(q_2, p_2^-)$ implies that $J(q_1, p_1^+) = J(q_2, p_2^+)$ but $J(q_1, p_1^+) \neq J(q_1, p_1^-)$.
- In polar coordinates, $J(r, p_r, \theta, p_\theta, \varphi, p_\varphi) = (r^2 p_\theta, p_\varphi)$
- Let (μ_1^-, μ_2^-) and (μ_1^+, μ_2^+) be the value of the momentum map before and after the impact, respectively.
- We can write $p_\theta^\pm = \mu_1^\pm / r^2$ and $p_\varphi^\pm = \mu_2^\pm$, so the reduced switching map is

$$p_r^- \mapsto (2\cos^2\theta - 1) p_r^- - 2r \sin\theta \cos\theta \frac{\mu_1}{r^2},$$

with the relations

$$\mu_1^+ = -\mu_1^-, \quad \mu_2^+ = \mu_2^-.$$

Rolling disk with dissipation hitting fixed walls IV

- The reduced switching surface can be written as

$$S_{(\mu_1,\mu_2)} = \Big\{(r, p_r) \big| r \sin\varphi = R \text{ or } r\sin\varphi = h - R, \\ \text{and } p_r \cos\varphi - r\frac{\mu_1}{r^2}\sin\varphi = R\mu_2, \\ \text{for some } \varphi \in [0, 2\pi)\Big\}\,.$$

Billiard with dissipation I

- Consider a particle in the plane which is free to move inside the surface defined by $x^2 + y^2 = 1$, i.e., $h(x, y) = x^2 + y^2 - 1$.
- The Hamiltonian of the system is $H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2)$, and the external force is $\alpha = 2c(p_x xy - p_y x^2)\mathrm{d}x - 2c(p_y xy - p_x y^2)\mathrm{d}y$.
- In polar coordinates, $H(\theta, r, p_\theta, p_r) = \frac{1}{2}(p_r^2 + r^2 p_\theta^2)$ and $\alpha(\theta, r, p_\theta, p_r) = -2cr^3 p_\theta \mathrm{d}r$.
- Clearly, $J(r, p_r, \theta, p_\theta) = r^2 p_\theta$ is preserved.
- Suppose that an elastic impact ($e = 1$) occurs at $S = \left\{x^2 + y^2 = 1,\ xp_x + yp_y \geq 0\right\}$.
- We conclude that $p_r^+ = -p_r^-$ and $p_\theta^+ = p_\theta^-$.

Billiard with dissipation II

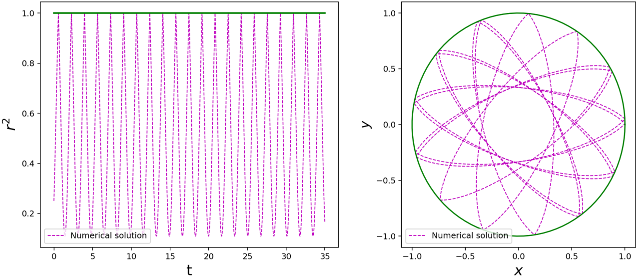

Figure: Simulation for $c = 2$. The first figure corresponds with the reduced trajectory while the second figure with the reconstructed solution.

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Billiard with dissipation III

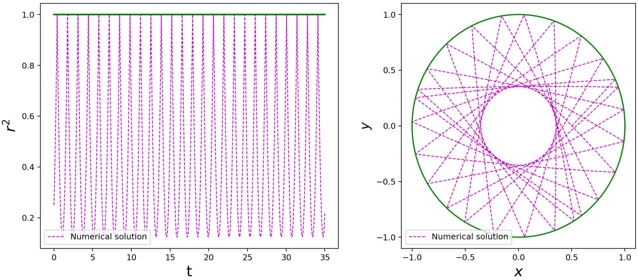

Figure: Simulation for $c = 0.20$. The first figure corresponds with the reduced trajectory while the second figure with the reconstructed solution.

Main references

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Thank you!

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