Integrability of contact Hamiltonian systems

A Liouville–Arnold theorem for contact dynamics

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Outline of the presentation

Theorem (Liouville–Arnold theorem)

Let $f_1, \ldots, f_n$ be independent functions in involution (i.e., $\{f_i, f_j\} = 0\ \forall i, j$) on a symplectic manifold (M^{2n}, ω). Let $M_\Lambda = \{x \in M \mid f_i = \Lambda_i\}$.

1. *Any compact connected component of M_Λ is diffeomorphic to $\mathbb{T}^n$.*
2. *On a neighborhood of M_Λ there are coordinates (φ^i, J_i) such that*

$$\omega = \mathrm{d}\varphi^i \wedge \mathrm{d}J_i,$$

and the Hamiltonian dynamics are given by

$$\frac{\mathrm{d}\varphi^i}{\mathrm{d}t} = \Omega^i, \qquad \frac{\mathrm{d}J_i}{\mathrm{d}t} = 0,$$

for some constants Ω^i.

Contact geometry

Definition

A (co-oriented) **contact manifold** is a pair (M, η), where M is an $(2n+1)$-dimensional manifold and η is a 1-form on M such that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form.

- The contact form η defines an isomorphism

$$\begin{aligned} \flat \colon \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto \iota_X \mathrm{d}\eta + \eta(X)\eta, \end{aligned}$$

- There exists a unique vector field $\mathcal{R}$ on (M, η), called the **Reeb vector field**, such that $\flat(\mathcal{R}) = \eta$, that is,

$$\iota_{\mathcal{R}} \mathrm{d}\eta = 0, \quad \iota_{\mathcal{R}} \eta = 1.$$

Contact geometry

- The **Hamiltonian vector field** of $f \in C^\infty(M)$ is given by

$$\flat(X_f) = \mathrm{d}f - (\mathcal{R}(f) + f)\,\eta,$$

- The **Jacobi bracket** is given by

$$\{f, g\} = -\mathrm{d}\eta(\flat^{-1}\mathrm{d}f, \flat^{-1}\mathrm{d}g) - f\mathcal{R}(g) + g\mathcal{R}(f).$$

- This bracket is bilinear and satisfies the Jacobi identity.
- However, unlike a Poisson bracket, it does not satisfy the Leibnitz identity:

$$\{f, gh\} \neq \{f, g\}h + \{f, h\}g.$$

Dissipated quantities

- In contact Hamiltonian dynamics dissipated quantities are akin to conserved quantities in symplectic dynamics.

Definition

An H-**dissipated quantity** is a function f on M such that

$$X_H(f) = -\mathcal{R}(H) f.$$

- A function f is H-dissipated iff

$$\{f, H\} = 0.$$

- Noether's theorem: symmetries $\leftrightarrow$ dissipated quantities.

- Let $M_{\langle\Lambda\rangle_+} = \{x \in M \mid \exists\, r \in \mathbb{R}^+ \colon f_\alpha(x) = r\Lambda_\alpha\}$.

Theorem (Colombo, de León, Lainz, L.-G., 2023)

Let (M, η) be a $(2n+1)$-dimensional contact manifold. Suppose that $f_0, f_1, \ldots, f_n$ are functions in involution such that $(\mathrm{d}f_\alpha)$ has rank at least n. Then, there is a neighborhood U of $M_{\langle\Lambda\rangle_+}$ such that

1. *$M_{\langle\Lambda\rangle_+}$ is invariant by the Hamiltonian flow of f_α and diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$.*
2. *There exists coordinates $(y^0, \ldots, y^n, \tilde{A}_1, \ldots, \tilde{A}_n)$ on U such that the equations of motion are given by*

$$\dot{y}^\alpha = \Omega^\alpha, \quad \dot{\tilde{A}}_i = 0.$$

3. *There exists a conformal transformation $\tilde{\eta} = \eta / A_0$ such that $(y^i, \tilde{A}_i, y^0)$ are Darboux coordinates for $(M, \tilde{\eta})$.*

Steps of the proof

1. Symplectize (M, η) and f_α, obtaining an exact symplectic manifold (M^Σ, θ) and homogeneous functions in involution f_α^Σ.
2. Prove a Liouville–Arnold theorem for exact symplectic manifolds with homogeneous functions in involution.
3. Un-symplectize the action-angle coordinates $(y_\Sigma^\alpha, A_\alpha^\Sigma)$ on M^Σ, yielding functions (y^α, A_Σ) on M.
4. Introduce action-angle coordinates $(y^\alpha, \tilde{A}_i)$ on M, where $\tilde{A}_i = -\frac{A_i}{A_0}$.

Exact symplectic manifolds: Liouville geometry

Definition

An **exact symplectic manifold** is a pair (M, θ), where M is a manifold and θ a one-form on N such that $\omega = -\mathrm{d}\theta$ is a symplectic form on M.

- The **Liouville vector field** Δ of (M, θ) is given by

$$\iota_\Delta \omega = -\theta.$$

- A tensor T is called **homogeneous of degree** n if $\mathcal{L}_\Delta T = nT$.

Symplectization of contact manifolds

Definition

Let (M, η) be a contact manifold. A **symplectization** is a fibre bundle $\Sigma\colon M^\Sigma \to M$, where (M^Σ, θ) is an exact symplectic manifold, such that

$$\sigma \Sigma^* \eta = \theta,$$

for a function σ on M^Σ called the **conformal factor**.

Symplectization of contact manifolds

- Contact geometry $\longleftrightarrow$ Liouville geometry
- Contact form η $\longleftrightarrow$ symplectic potential θ
- Functions $\longleftrightarrow$ homogeneous functions of degree 1
- Hamiltonian vector fields $\longleftrightarrow$ Hamiltonian vector fields, homogeneous of degree 0

Symplectization of contact manifolds

Theorem

Given a symplectization $\Sigma\colon (M^\Sigma, \theta) \to (M, \eta)$ with conformal factor σ, there is a bijection between functions f on M and homogeneous functions of degree 1 f^Σ on M^Σ such that

$$\Sigma_* (X_{f^\Sigma}) = X_f .$$

This bijection is given by

$$f^\Sigma = \sigma \Sigma^* f .$$

Moreover, one has

$$\left\{f^\Sigma, g^\Sigma\right\}_\theta = \{f, g\}_\eta^\Sigma .$$

Symplectization of contact manifolds

Example

$\Sigma = \pi_1 \colon (M \times \mathbb{R}^+, \theta = r\eta) \to (M, \eta)$ is a symplectization with conformal factor $\sigma = r$, for r the global coordinate on $\mathbb{R}^+$.

Liouville–Arnold theorem for exact symplectic manifolds

- We want to obtain action-angle coordinates $(\varphi^{\alpha}_{\Sigma}, J^{\Sigma}_{\alpha})$ on (M^{Σ}, θ) in order to define functions $(\varphi^{\alpha}, J_{\alpha})$ on (M, η)
- We need homogeneous objects on (M^{Σ}, θ) so that they have a correspondence with objects on (M, η).
- However, the classical Liouville–Arnold theorem does not take into account the homogeneity of θ and f^{Σ}_{α}.
- Moreover, we need to consider non-compact level sets of f^{Σ}_{α}.

Liouville–Arnold theorem for exact symplectic manifolds

Theorem (Colombo, de León, Lainz, L.-G., 2023)

Let (M, θ) be an exact symplectic manifold. Suppose that the functions f_α, $\alpha = 1, \ldots, n$, on M are independent, in involution and homogeneous of degree 1. Let U be an open neighborhood of M_Λ such that:

1. *f_α have no critical points in U,*
2. *the Hamiltonian vector fields of X_{f_α} are complete,*
3. *the submersion $(f_\alpha)\colon U \to \mathbb{R}^n$ is a trivial bundle over $V \subseteq \mathbb{R}^n$.*

Then, $U \simeq \mathbb{R}^{n-m} \times \mathbb{T}^m \times V$, provided with action-angle coordinates (y^α, A_α) such that

$$\theta = A_\alpha \mathrm{d}y^\alpha, \qquad \frac{\mathrm{d}y^\alpha}{\mathrm{d}t} = \Omega^\alpha, \qquad \frac{\mathrm{d}A_\alpha}{\mathrm{d}t} = 0.$$

Sketch of proof

- Since X_{f_α} are n vector fields tangent to M_Λ, linearly independent and pairwise commutative, they generate the algebra $\mathbb{R}^n$ and $M_\Lambda \simeq \mathbb{R}^n/\mathbb{Z}^k$.
- Thus there are coordinates $y^\alpha = M_\alpha^\beta s^\beta$, where $X_{f_\alpha}(s^\beta) = \delta_\alpha^\beta$.
- The values of f_α define coordinates (J_α) on V.
- Since M_Λ is Lagrangian, $\theta = A_\alpha(J)\mathrm{d}y^\alpha + B^\alpha(y, J)\mathrm{d}J_\alpha$.
- Since f_α are homogeneous of degree 1, $\theta(X_{f_\alpha}) = f_\alpha$.
- By construction, $\Delta(y^\alpha) = 0$.
- With additional contractions with θ and ω, one concludes that $\theta = A_\alpha \mathrm{d}y^\alpha$, where $J_\beta = M_\beta^\alpha J_\alpha$.

From conditions on f_α^Σ to conditions on f_α

- In order to apply the Liouville–Arnold theorem for exact symplectic manifolds, we need to translate the conditions on f_α to conditions on f_α^Σ.
- Let $M_{\langle\Lambda\rangle_+} = \{x \in M \mid \exists\, r \in \mathbb{R}^+ \colon F(x) = r\Lambda\}$.
- Let $F^\Sigma = (f_\alpha^\Sigma)$ and $\tilde{M}_\Lambda = (F^\Sigma)^{-1}(\Lambda)$.
- Given the functions $f_0, f_1, \ldots, f_n \colon M \to \mathbb{R}$, let $F = (f_\alpha)$ and

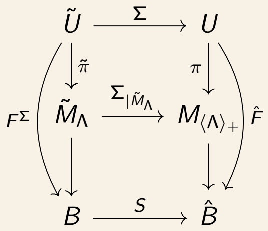
$$\hat{F} = S \circ F \colon M \to \mathbb{S}^n,$$

where $S \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{S}^n$ denotes the projection on the sphere.

From conditions on f_α^Σ to conditions on f_α

Lemma

Given $\langle\Lambda\rangle_+ \in S^n$, let $\hat{B} \subseteq S^n$ be an open neighborhood of $\langle\Lambda\rangle_+$ and let $\pi\colon U \to M_{\langle\Lambda\rangle_+}$ be a tubular neighborhood of $M_{\langle\Lambda\rangle_+}$ such that $\hat{F}_{|U}\colon U \to \hat{B}$ is a submersion with diffeomorphic fibers. Define $B = S^{-1}(\hat{B})$ and $\tilde{U} = \Sigma^{-1}(U)$ and $\tilde{\pi} = \Sigma_{|\tilde{M}_\Lambda}^{-1} \circ \pi \circ \Sigma$. Then, $\tilde{\pi}\colon \tilde{U} \to \tilde{M}_\Lambda$ is a tubular neighborhood of $\tilde{M}_\Lambda$ such that $F^\Sigma_{|\tilde{U}}\colon \tilde{U} \to B$ is a submersion with diffeomorphic fibers.



Theorem (Colombo, de León, Lainz, L.-G., 2023)

Let (M, η) be a $(2n+1)$-dimensional contact manifold. Suppose that $f_0, f_1, \ldots, f_n$ are functions in involution such that $(\mathrm{d} f_\alpha)$ has rank at least n. Assume that the Hamiltonian vector fields X_{f_α} are complete. Given $\Lambda \in \mathbb{R}^{n+1} \setminus \{0\}$, let $\hat{B} \subseteq S^n$ be an open neighborhood of $\langle \Lambda \rangle_+$ and let $\pi \colon U \to M_{\langle \Lambda \rangle_+}$ be a tubular neighborhood of $M_{\langle \Lambda \rangle_+}$ such that $\hat{F}_{|U} \colon U \to \hat{B}$ is a submersion with diffeomorphic fibers. Then

1. *$M_{\langle \Lambda \rangle_+}$ is invariant by the Hamiltonian flow of f_α and diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$.*
2. *There exists coordinates $(y^0, \ldots, y^n, \tilde{A}_1, \ldots, \tilde{A}_n)$ on U such that the equations of motion are given by*

$$\dot{y}^\alpha = \Omega^\alpha, \quad \dot{\tilde{A}}_i = 0.$$

3. *There exists a conformal transformation $\tilde{\eta} = \eta / A_0$ such that $(y^i, \tilde{A}_i, y^0)$ are Darboux coordinates for $(M, \tilde{\eta})$.*

Sketch of the proof

1. Symplectize (M, η) and f_α, in order to apply the Liouville–Arnold theorem for exact symplectic manifolds
    - $\{f_\alpha, f_\beta\} = 0 \Rightarrow \ \{f_\alpha^\Sigma, f_\beta^\Sigma\} = 0$.
    - X_{f_α} complete $\Rightarrow \ X_{f_\alpha^\Sigma}$ complete.
    - $\operatorname{rank} \mathrm{d} f_\alpha \geq n \Rightarrow \ \operatorname{rank} \mathrm{d}(\underbrace{\sigma \Sigma^* f_\alpha}_{f_\alpha^\Sigma}) \geq n+1$.
    - $\Sigma\left((F^\Sigma)^{-1}(\Lambda)\right) = \left\{x \in M \mid \exists s \in \mathbb{R}^+ : \ F(x) = \frac{\Lambda}{s}\right\} = M_{\langle\Lambda\rangle_+}$.
    - X_{f_α} commute and are tangent to $M_{\langle\Lambda\rangle_+} \Rightarrow M_{\langle\Lambda\rangle_+} \simeq \mathbb{T}^k \times \mathbb{R}^{n+1-k}$.
2. Un-symplectize the action-angle coordinates $(y_\Sigma^\alpha, A_\alpha^\Sigma)$ on $\tilde{U}$, yielding functions (y^α, A_α) on U.
3. Introduce action-angle coordinates $(y^\alpha, \tilde{A}_i)$ on U
    - Since $\Lambda \neq 0$, $\exists A_\alpha \neq 0$. W.l.o.g., assume $A_0 \neq 0$.
    - Then $\left(y^\alpha, \tilde{A}_i = -\frac{A_i}{A_0}\right)$ are coordinates on U.

Sketch of the proof

- By construction, y^α are linear combinations of flows of X_{f_α}, namely,

$$X_{f_\alpha} = M^\alpha_\beta \frac{\partial}{\partial s^\beta}.$$

- Therefore, the dynamics are given by

$$\frac{\mathrm{d}y^\alpha}{\mathrm{d}t} = \Omega^\alpha, \qquad \frac{\mathrm{d}\tilde{A}_i}{\mathrm{d}t} = 0.$$

- $\theta^\Sigma = A^\Sigma_\alpha \mathrm{d}y^\alpha_\Sigma \rightsquigarrow \eta = A_\alpha \mathrm{d}y^\alpha$, so

$$\tilde{\eta} = \frac{1}{A_0}\eta = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i.$$

Other notions of integrability

- Khesin and Tabachnikov, Liberman, Banyaga and Molino, Lerman, etc. have defined notions of contact complete integrability which are geometric but not dynamical, e.g. a certain foliation over a contact manifold.
- Boyer considers the so-called good Hamiltonians H, i.e., $\mathcal{R}(H) = 0 \rightsquigarrow$ no dissipated quantities, "symplectic" dynamics.
- We are interested in complete integrability of contact Hamiltonian dynamics.

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¡Gracias por vuestra atención!

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