Symmetries, conservation and dissipation in time-dependent contact systems

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Outline of the presentation

Motivation

- Since the seminal work by Emmmy Noether, the relation between symmetries and conserved quantities has been fundamental in mathematical/theoretical physics.
- If one cannot solve a nonlinear system explicitly, at least knowing its symmetries can provide a qualitative description of its behaviour.
- Reduction procedures can be used in order to simplify the description of a dynamical system whose group of symmetries is known.

Review on symmetries for symplectic mechanics

Symplectic geometry is the natural framework for time-independent classical mechanics.

Theorem

Consider a Hamiltonian system (M, ω, H). Let $Y \in \mathfrak{X}(M)$. If the flow of Y is a symplectomorphism ($\mathcal{L}_Y \omega = 0$) and preserves the Hamiltonian function ($\mathcal{L}_Y H = 0$), then the local functions $f \colon U \subset M \to \mathbb{R}$ given by

$$\iota_Y \omega = \mathrm{d}f$$

are constants of the motion.

The proof is an easy exercise of Cartan calculus.

Review on symmetries for symplectic mechanics

Example (Energy)

We have that $\mathcal{L}_{X_H}\omega = 0$ and $\mathcal{L}_{X_H}H = 0$, so H is a conserved quantity. (This is no longer the case if H depends explicitly on time.)

Example (Linear momentum)

Suppose that $M = \mathrm{T}^*\mathbb{R} \simeq \mathbb{R}^2,\ \omega = \mathrm{d}q \wedge \mathrm{d}p$ and $H = \dfrac{p^2}{2}$. One can easily check that $Y = \dfrac{\partial}{\partial q}$ verifies $\mathcal{L}_Y\omega = 0$ and $\mathcal{L}_Y H = 0$, so $f = p$ is conserved.

A quite complete and accessible reference is
N. Román-Roy, "A summary on symmetries and conserved quantities of autonomous Hamiltonian systems," **J. Geom. Mech.**, 2020.

Cosymplectic and contact structures

Let M be a $(2n+1)$-dimensional manifold

Cosymplectic manifold (M, ω, τ)

- ω closed 2-form
- τ closed 1-form
- $\tau \wedge \omega^n \neq 0$
- Reeb vector field $\mathcal{R}_t$:

$$\iota_{\mathcal{R}_t}\omega = 0, \ \iota_{\mathcal{R}_t}\tau = 1$$

- Darboux coords. (t, q^i, p_i):

$$\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i, \ \tau = \mathrm{d}t, \ \mathcal{R}_t = \frac{\partial}{\partial t}$$

Contact manifold (M, η)

- η 1-form
- $\eta \wedge \mathrm{d}\eta^n \neq 0$
- Reeb vector field $\mathcal{R}_t$:

$$\iota_{\mathcal{R}_t}\eta = 1, \quad \iota_{\mathcal{R}_t}\mathrm{d}\eta = 0$$

- Darboux coords. (q^i, p_i, z):

$$\eta = \mathrm{d}z - p_i\mathrm{d}q^i, \ \mathcal{R}_z = \frac{\partial}{\partial z}$$

Cocontact structures

- Idea: a structure that combines the cosymplectic and contact ones.

Definition

A **cocontact manifold** is a triple (M, τ, η) where:

1. M is a $(2n+2)$-dimensional manifold,
2. τ and η are 1-forms,
3. $\mathrm{d}\tau = 0$,
4. $\tau \wedge \eta \wedge (\mathrm{d}\eta)^{\wedge n} \neq 0$.

Cocontact structures

- Given a cocontact manifold (M, τ, η), we have the **flat isomorphism**:

$$\begin{aligned} \flat \colon \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto (\iota_X \tau)\tau + \iota_X \mathrm{d}\eta + (\iota_X \eta)\,\eta \end{aligned}$$

and its inverse $\sharp = \flat^{-1}$.

- **Reeb vector fields**: $\mathcal{R}_t = \flat^{-1}(\tau),\ \mathcal{R}_z = \flat^{-1}(\eta)$.
- Darboux coordinates $\left(t, q^i, p_i, z\right)$:

$$\tau = \mathrm{d}t, \quad \eta = \mathrm{d}z - p_i \mathrm{d}q^i, \quad \mathcal{R}_t = \frac{\partial}{\partial t}, \quad \mathcal{R}_z = \frac{\partial}{\partial z}$$

Cocontact Hamiltonian systems

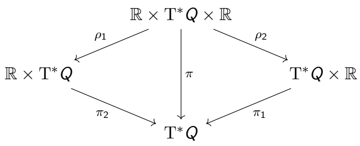
- Given a Hamiltonian function $H\colon M \to \mathbb{R}$, its **Hamiltonian vector field** is given by

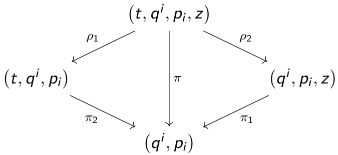
$$\flat(X_H) = \mathrm{d}H - (\mathcal{R}_z(H) + H)\,\eta + (1 - \mathcal{R}_t(H))\,\tau\,.$$

- In Darboux coordinates,

$$X_H = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i\frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z}\,.$$

Canonical cocontact manifold





- Let Q be an n-dimensional manifold with local coordinates (q^i).
- Let $\theta_0 = p_i \mathrm{d}q^i$ be the canonical 1-form of T^*Q.
- Consider the 1-forms $\theta_Q = \pi^*\theta_0$ and $\eta_Q = \mathrm{d}z - \theta_Q$ on $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$
- Then, $(\mathrm{d}t, \eta_Q)$ is a cocontact structure on $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$. The local expression of the 1-form η is

$$\eta_Q = \mathrm{d}z - p_i \mathrm{d}q^i \,.$$

Dissipated quantities

- Given a (time-independent) contact Hamiltonian system (M, η, H), we have
$$X_H(H) = -\mathcal{R}_z(H)H.$$
- A similar behavior is observed in other quantities which are conserved for symplectic Hamiltonian systems.

Example (Linear momentum)

Let $M = \mathbb{R}^4$ and $H = \frac{p^2}{2} - \gamma(t)z$. Then,
$$X_H(p) = -\gamma(t)p.$$

Dissipated quantities

- This motivates the following:

Definition

Let (M, τ, η, H) be a cocontact Hamiltonian system. A **dissipated quantity** is a function $f\colon M \to \mathbb{R}$ such that

$$X_H(f) = -\mathcal{R}_z(H) f \,.$$

Theorem (Noether's theorem)

Consider the cocontact Hamiltonian system (M, τ, η, H). Let $Y \in \mathfrak{X}(M)$.

1. *If $\eta([Y, X_H]) = 0$ and $\tau(Y) = 0$, then $f = -\eta(Y)$ is a dissipated quantity.*
2. *Conversely, given a dissipated quantity f, the vector field $Y = X_f - \mathcal{R}_t$ verifies $\eta([Y, X_H]) = 0$, $\tau(Y) = 0$ and $f = -\eta(Y)$.*

Definition

A **generalized infinitesimal dynamical symmetry** is a vector field $Y \in \mathfrak{X}(M)$ such that $\eta([Y, X_H]) = 0$ and $\tau(Y) = 0$.

- We can consider symmetries which preserve the Hamiltonian vector field (and hence map integral curves into integral curves).

Definition

Let (M, τ, η, H) be a cocontact Hamiltonian system and let X_H be its cocontact Hamiltonian vector field.

1. An **infinitesimal dynamical symmetry** is a vector field $Y \in \mathfrak{X}(M)$ such that $\mathcal{L}_Y X_H = 0$ and $\iota_Y \tau = 0$.
2. If $M = \mathbb{R} \times N$ with N a contact manifold, a **dynamical symmetry** is a diffeomorphism $\Phi \colon M \to M$ such that $\Phi_* X_H = X_H$ and $\Phi^* t = t$.

- If $\sigma \colon \mathbb{R} \to M$ is an integral curve of X_H and Φ is a dynamical symmetry, then $\Phi \circ \sigma$ is also an integral curve of X_H.

Definition

An **infinitesimal ρ-conformal cocontactomorphism** is a vector field $Y \in \mathfrak{X}(M)$ such that $\mathcal{L}_Y \eta = \rho\eta$ and $\mathcal{L}_Y \tau = \tau$ for some $\rho \colon M \to \mathbb{R}$.

Proposition

An infinitesimal ρ-conformal cocontactomorphism Y is a generalized infinitesimal dynamical symmetry if, and only if, $\mathcal{L}_Y H = \rho H$ and $\iota_Y \tau = 0$. If this holds, Y is called an ***infinitesimal ρ-conformal Hamiltonian symmetry***

- We can consider the following generalization of infinitesimal ρ-conformal Hamiltonian symmetries:

Definition

Given a cocontact Hamiltonian system (M, τ, η, H), a (ρ, g)-**Cartan symmetry** is a vector field $Y \in \mathfrak{X}(M)$ such that

$$\mathcal{L}_Y \eta = \rho\eta + \mathrm{d}g\,, \qquad \mathcal{L}_Y H = \rho H + g\mathcal{R}_z(H)\,, \qquad \iota_Y \tau = 0\,.$$

Theorem

If Y is a (ρ, g)-Cartan symmetry, then $f = g - \iota_Y \eta$ is a dissipated quantity.

Classification of infinitesimal symmetries

Generalized infinitesimal dynamical symmetries

$\tau(Y) = 0 \qquad \eta([Y, X_H]) = 0$

Infinitesimal dynamical symmetries

$\tau(Y) = 0$

$[Y, X_H] = 0$

Infinitesimal conformal Hamiltonian symmetries

$\tau(Y) = 0 \quad \mathcal{L}_Y \eta = \rho\eta$

$Y(H) = \rho H$

Infinitesimal conformal cocontactomorphisms

$\tau(Y) = 0$

$\mathcal{L}_Y \eta = \rho\eta$

Infinitesimal strict Hamiltonian symmetries

$\tau(Y) = 0 \quad \mathcal{L}_Y \eta = 0$

$Y(H) = 0$

Cartan symmetries

$\tau(Y) = 0 \quad \mathcal{L}_Y \eta = \rho\eta + \mathrm{d}g$

$Y(H) = \rho H + g\mathcal{R}_z(H)$

Lie algebras and Lie groups of symmetries

Proposition

1. *If Y_1 and Y_2 are infinitesimal dynamical symmetries, then $[Y_1, Y_2]$ is also an infinitesimal dynamical symmetry.*
2. *If Φ_1 and Φ_2 are dynamical symmetries, then $\Phi_1 \circ \Phi_2$ is also a dynamical symmetry.*
3. *If Y_a is a ρ_a-conformal Hamiltonian symmetry ($a = 1, 2$), then $[Y, Z]$ is a $\widetilde{\rho}$-conformal Hamiltonian symmetry, where $\widetilde{\rho} = Y_1(\rho_2) - Y_2(\rho_1)$.*

There are counterexamples showing that neither generalized infinitesimal dynamical symmetries nor Cartan symmetries close Lie subalgebras.

Lagrangian formalism

- Given a smooth n-dimensional manifold Q, consider the product manifold $\mathbb{R} \times \mathrm{T}Q \times \mathbb{R}$ equipped with adapted coordinates (t, q^i, v^i, z)
- Consider a Lagrangian function $L\colon \mathbb{R} \times \mathrm{T}Q \times \mathbb{R} \to \mathbb{R}$. Hereinafter, assume L to be regular, i.e., the Hessian matrix

$$(W_{ij}) = \left(\frac{\partial^2 L}{\partial v^i \partial v^j} \right)$$

is non-singular.

- The dynamics are given by the **Herglotz–Euler–Lagrange equations**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial z}\frac{\partial L}{\partial v^i}\,, \qquad \dot{z} = L\,.$$

Lagrangian formalism

- If L is regular, then $(\mathbb{R} \times \mathrm{T}Q \times \mathbb{R}, \mathrm{d}t, \eta_L, E_L)$ is a cocontact Hamiltonian system.
- The Lagrangian energy and the contact form are locally given by

$$E_L = v^i \frac{\partial L}{\partial v^i} - L\,, \qquad \eta_L = \mathrm{d}z - \frac{\partial L}{\partial v^i}\,\mathrm{d}q^i\,,$$

- The Reeb vector fields are locally

$$\mathcal{R}_t^L = \frac{\partial}{\partial t} - W^{ij}\frac{\partial^2 L}{\partial t \partial v^j}\frac{\partial}{\partial v^i}\,, \qquad \mathcal{R}_z^L = \frac{\partial}{\partial z} - W^{ij}\frac{\partial^2 L}{\partial z \partial v^j}\frac{\partial}{\partial v^i}\,,$$

where (W^{ij}) is the inverse of the Hessian matrix (W_{ij}).

Cyclic coordinates

Suppose that $\frac{\partial L}{\partial q^1} = 0$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial z}\frac{\partial L}{\partial v^i}$$

implies that

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = \frac{\partial L}{\partial z} p_1,$$

where $p_i := \frac{\partial L}{\partial v^i}$.
Hence, along the trajectories $(q^i(t), v^i(t), z(t))$,

$$p_1(t) = p_1(0) \exp\left(\int_0^t \frac{\partial L}{\partial z}(q^i(s), v^i(s), z(s)) \, \mathrm{d}s\right)$$

Cyclic coordinates

Example

Consider a Lagrangian function of the form

$$L = \frac{1}{2} g_{ij} v^i v^j - V(t, q^2, q^3, \ldots, q^n) - \kappa z,$$

for some constant κ.
Then, q^1 is a cyclic coordinate. Thus,

$$\dot{p}_1 = \frac{\partial L}{\partial z} p_1 = -\kappa p_1,$$

so

$$p_1(t) = p_1(0) e^{-\kappa t}$$

Symmetries of the Lagrangian

Given $Y \in \mathfrak{X}(Q)$, we define $Y^C,\ Y^V \in \mathfrak{X}(\mathbb{R} \times \mathrm{T}Q \times \mathbb{R})$. Locally,

$$Y = Y^i \frac{\partial}{\partial q^i}, \qquad Y^V = Y^i \frac{\partial}{\partial v^i}, \qquad Y^C = Y^i \frac{\partial}{\partial q^i} + v^i \frac{\partial Y^i}{\partial q^j} \frac{\partial}{\partial v^i}.$$

Theorem

Let $Y \in \mathfrak{X}(Q)$. Then $Y^C(L) = 0$ iff $Y^V(L)$ is a dissipated quantity. If this holds, then Y^C is called an ***infinitesimal natural symmetry of the Lagrangian***

Proposition

Infinitesimal natural symmetries of the Lagrangian form a Lie subalgebra of $(\mathfrak{X}(\mathbb{R} \times \mathrm{T}Q \times \mathbb{R}), [\cdot, \cdot])$.

Proposition

An vector field $Z \in \mathfrak{X}(\mathbb{R} \times \mathrm{T}Q \times \mathbb{R})$ with local expression

$$Z = \zeta(t, q, v, z) \frac{\partial}{\partial z}$$

is a generalized infinitesimal dynamical symmetry iff ζ is a dissipated quantity.
If this is the case, we call Z an ***infinitesimal action symmetry****.*

The free particle with time-dependent mass and linear dissipation

Consider the cocontact Hamiltonian system $(\mathbb{R} \times \mathrm{T}^*\mathbb{R} \times \mathbb{R}, \mathrm{d}t, \eta, H)$, where

$$H = \frac{p^2}{2m(t)} + \frac{\kappa}{m(t)} z\,,$$

with m a function depending only on t, expressing the mass of the particle, and κ a positive constant. The Hamiltonian vector field of H is

$$X_H = \frac{\partial}{\partial t} + \frac{p}{m(t)} \frac{\partial}{\partial q} - p \frac{\kappa}{m(t)} \frac{\partial}{\partial p} + \left(\frac{p^2}{2m(t)} - \frac{\kappa}{m(t)} z \right) \frac{\partial}{\partial z}\,.$$

The free particle with time-dependent mass and linear dissipation

The function

$$f(t, q, p, z) = \exp\left(-\int_0^t \frac{\kappa}{m(s)} \mathrm{d}s\right)$$

is a dissipated quantity. Hence, by Noether's Theorem, the vector field

$$Y_f = X_f - \mathcal{R}_t = -\exp\left(-\int_0^t \frac{\kappa}{m(s)} \mathrm{d}s\right) \frac{\partial}{\partial z}$$

is a generalized infinitesimal dynamical symmetry.

The free particle with time-dependent mass and linear dissipation

In addition, one can verify that Y_f is an infinitesimal dynamical symmetry, namely $[Y_f, X_H = 0]$.
Moreover,

$$Y_f(H) = -\exp\left(-\int_0^t \frac{\kappa}{m(s)}\mathrm{d}s\right)\mathcal{R}_z(H)\,,$$

and

$$\mathcal{L}_{Y_f}\eta = -\mathrm{d}\left(\exp\left(-\int_0^t \frac{\kappa}{m(s)}\mathrm{d}s\right)\right)\,,$$

so Y_f is a $(0, g)$-Cartan symmetry, where $g = -\exp\left(-\int_0^t \frac{\kappa}{m(s)}\mathrm{d}s\right)$.

The free particle with time-dependent mass and linear dissipation

The Lagrangian counterpart of this system is characterized by the Lagrangian function $L\colon \mathbb{R} \times \mathrm{T}\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$L = m(t)\frac{v^2}{2} - \frac{\kappa}{m(t)} z\,.$$

The vector field $Z \in \mathfrak{X}(\mathbb{R} \times \mathrm{T}\mathbb{R} \times \mathbb{R})$ with local expression

$$Z = \zeta \frac{\partial}{\partial z}, \qquad \zeta(t, q, v, z) = \exp\left(-\int_0^t \frac{\kappa}{m(s)} \mathrm{d}s\right)$$

is an infinitesimal action symmetry, since ζ is a dissipated quantity.

An action-dependent central potential with time-dependent mass

Consider a Lagrangian function $L\colon \mathbb{R} \times \mathrm{T}\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ of the form

$$L = \frac{m(t)}{2}\left(v_x^2 + v_y^2\right) - V\left(t, (x^2+y^2), z\right) ,$$

where $m(t)$ is a positive-valued function. Let $Y \in \mathfrak{X}(\mathbb{R}^2)$ be infinitesimal generator of rotations on the plane, namely,

$$Y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} .$$

Then,

$$\bar{Y}^C = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v_y\frac{\partial}{\partial v_x} + v_x\frac{\partial}{\partial v_y} , \qquad \bar{Y}^V = -y\frac{\partial}{\partial v_x} + x\frac{\partial}{\partial v_y} .$$

An action-dependent central potential with time-dependent mass

Clearly, $\bar{Y}^C$ is an infinitesimal natural symmetry of the Lagrangian, i.e.,

$$\bar{Y}^C(L) = 0.$$

Hence,

$$\bar{Y}^V(L) = m(t)\left(-yv_x + xv_y\right)$$

is a dissipated quantity.
This quantity is the angular momentum for a particle with time-dependent mass.

The two-body problem with time-dependent friction

- The phase space is $\mathbb{R} \times \mathrm{T}\mathbb{R}^6 \times \mathbb{R}$, with coords. $(t, \mathbf{q}^1, \mathbf{q}^2, \mathbf{v}^1, \mathbf{v}^2, z)$.
- The superindex denotes each particle, and the bold notation is a shorthand for the three spatial components.
- The Lagrangian function is

$$L = \frac{1}{2} m_1 \mathbf{v}^1 \cdot \mathbf{v}^1 + \frac{1}{2} m_2 \mathbf{v}^2 \cdot \mathbf{v}^2 - U(r) - \gamma(t) z \,,$$

  where $m_1, m_2 \in \mathbb{R}$ are the masses of the particles, $U(r)$ is the central potential and γ is a time-dependent function.
- Consider the vector fields

$$Y_i = \frac{1}{m_1 + m_2} \left(\frac{\partial}{\partial q_i^1} + \frac{\partial}{\partial q_i^2} \right) \quad i = 1, 2, 3.$$

The two-body problem with time-dependent friction

- Then,

$$Y_i^C = \frac{1}{m_1 + m_2}\left(\frac{\partial}{\partial q_i^1} + \frac{\partial}{\partial q_i^2}\right), \quad i = 1, 2, 3,$$

  and $Y_i^C(L) = 0$, so they are infinitesimal natural symmetries of the Lagrangian.
- The associated dissipated quantities are

$$Y_i^V(L) = \frac{m_1 v_i^1 + m_2 v_i^2}{m_1 + m_2}, \quad i = 1, 2, 3.$$

The two-body problem with time-dependent friction

- The center of masses is given by

$$\mathbf{R} = \frac{m_1 \mathbf{q}^1 + m_2 \mathbf{q}^2}{m_1 + m_2}.$$

so

$$\dot{\mathbf{R}} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \frac{m_1 \mathbf{v}^1 + m_2 \mathbf{v}^2}{m_1 + m_2} = \left(Y_1^V(L), Y_2^V(L), Y_3^V(L)\right)$$

is made up of 3 dissipated quantities.

- Along a solution, it evolves as

$$\dot{\mathbf{R}}(t) = \dot{\mathbf{R}}_0 e^{-\int \gamma(t) \mathrm{d}t}.$$

In particular, if γ is a positive constant, as the time increases the center of mass tends to move on a line with constant speed $\dot{\mathbf{R}}_0$.

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Thank you!

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