Homogeneous Darboux and Frobenius theorems

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The interest of gradings

There are several scenarios in geometry and physics in which a $(\mathbb{N}, \mathbb{Z}, \mathbb{Z}_2, \mathbb{R} \ \ldots)$ grading appears:

- the algebra of exterior forms with the exterior product $\big(\Omega^\bullet(M), \wedge\big)$,
- the spin of particles,
- intensive/extensive variables in thermodynamics,
- symplectisation/Poissonisation of contact/Jacobi manifolds,
- supermanifolds,
- higher tangent bundles.

Theorem (Euler)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, and k an integer. The following assertions are equivalent:

1. $f(t \cdot x) = t^k f(x)\,, \quad \forall\, t \in \mathbb{R} \setminus \{0\}\,, \forall\, x \in \mathbb{R}^n\,,$
2. *f is an eigenfunction of $X = \sum_{i=1}^{n} x^i \partial_{x^i}$ with eigenvalue k, namely*

$$X(f) = k \cdot f\,.$$

Definition

A function f satisfying any of the equivalent conditions above is called **homogeneous of degree** k or k-**homogeneous**.

We can extend this notion to a manifold M^n by considering a vector field $X \in \mathfrak{X}(M)$ which is locally of the form

$$X = \sum_{i=1}^{n} x^i \partial_{x^i}$$

in a certain atlas.

Definition

An (even) vector field ∇ on a (super)manifold M is called a **weight vector field** if in a neighbourhood of every point of (the body of) M there are coordinates (x^a) such that

$$\nabla = \sum_{a=1}^{n} w_a \cdot x^a \partial_{x^a}, \quad w_a \in \mathbb{R}.$$

Such coordinates are called **homogeneous coordinates**, and the pair (M, ∇) is called a **homogeneity (super)manifold**.

Definition

Let (M, ∇) be a homogeneity (super)manifold and $w \in \mathbb{R}$. A tensor field A on M is called **homogeneous of degree** w or w-**homogeneous** if

$$\mathcal{L}_\nabla A = w \cdot A\,.$$

Example (Trivial)

The zero-section of the tangent bundle makes any (super)manifold a homogeneity (super)manifold:

$$\nabla \equiv 0\,.$$

This means that all the subsequent results I shall present still hold if you forget the adjective "homogeneous".

Example (Vector bundles)

Let $\pi \colon E \to M$ be a vector bundle (VB). The Euler vector field $\nabla_E \in \mathfrak{X}(E)$, i.e. the infinitesimal generator of homotheties on the fibers, is a weight vector field. In bundle coordinates,

$$\pi \colon (x^i, y^a) \mapsto (x^i)\,, \quad \nabla_E = \sum_a y^a \partial_{y^a}\,.$$

Remark

The structure of VB on E is uniquely determined[a] by its structure of manifold and a smooth action of the monoid $(\mathbb{R}, \cdot)$ generated by ∇_E

[a] See Grabowski and Rotkiewicz, *J. Geom. Phys.* 59 (2009).

Example (Exact symplectic manifolds)

Let (M, ω) be a symplectic manifold. Then, the following statements are equivalent:

1. ω is exact, i.e. there exists a $\theta \in \Omega^1(M)$ such that $\omega = \mathrm{d}\theta$,
2. there exists a **Liouville vector field** $\nabla \in \mathfrak{X}(M)$ such that $\mathcal{L}_\nabla \omega = \omega$.

In fact, since $\mathfrak{X}(M) \ni X \mapsto \iota_X \omega \in \Omega^1(M)$ is an isomorphism, given θ (resp. ∇), we can univocally define θ (resp. ∇) by the relation

$$\iota_\nabla \omega = \theta\,.$$

The Liouville vector field is a weight vector field. Indeed, in Darboux coordinates (q^i, p_i) for θ, we have

$$\theta = p_i \mathrm{d}q^i \Longrightarrow \nabla = p_i \partial_{p_i}\,.$$

Example (Blackhole thermodynamics)

In thermodynamics, one can regard extensive (resp. intensive) variables as homogeneous of degree 1 (resp. 0).

"When gravity plays an important role the standard extensivity property of thermodynamics does not hold; nevertheless, one can still find a form of thermodynamics in which each thermodynamic variable follows a power scaling law where the power can be different from one or zero, i.e., the variables are allowed to be neither extensive nor intensive. Black hole thermodynamics is the most evident and special thermodynamics belonging to this framework." Belgiorno, J. Math. Phys. **44** (2003).

In this context, weight vector fields with non-integer weights appear.

Given a homogeneity manifold (M, ∇) and an open subset $U \subseteq M$, it is important to distinguish two possible situations:

1. $\nabla|_U$ is nowhere zero,
2. there exists a point $x_0 \in M$ such that $\nabla(x_0) = 0$.

Proposition

Any nowhere-vanishing vector field X on a manifold M^n is a weight vector field. However, its weights are not canonical.

Proof.

Since X is nowhere zero, there exists an atlas with local coordinates (x^a) such that $X = \partial_{x^1}$. For any $\Gamma := \{w_1, \dots, w_n\} \subset \mathbb{R}$ with $w_1 \neq 0$, we can define a new system of coordinates

$$y^1 = e^{w_1 x^1}\,, \quad y^i = e^{w_i x^1} x^i\,, \quad 2 \leq i \leq n\,,$$

so that

$$X = \sum_{a=1}^{n} w_a \cdot y^a \partial_{y^a}\,.$$

$\square$

On the other hand, in a neighbourhood of any point at which the weight vector field vanishes, its weights are canonical.

Proposition (Grabowska and Grabowski, 2024)

An (even) vector field $\nabla \in \mathfrak{X}(M)$ is a weight vector field iff $\mathrm{T}_{x_0} M$ is diagonal at any $x_0 \in |M|$ such that $\nabla(x_0) = 0$. In that case, the weights of any system of homogeneous coordinates around x_0 are the eigenvalues $w_1, \ldots, w_n$ of the matrix $(\mathrm{T}_{x_0} M)$.

A supercrash course on supergeometry

Remark

I will not provide rigurous definitions of supermanifold, super differential forms, etc. The next couple of slides merely intend to provide the main ideas that I will be mentioning during my talk for those in the audience unfamiliar[a] with the topic.

[a] Those of you who are well-acquainted with the topic, please accept my superapologies.

A supercrash course on supergeometry

- The superspace $\mathbb{R}^{p|q}$ has canonical coordinates $(x^1, \ldots, x^p, \xi^q, \ldots, \xi^q)$, where x are commuting and ξ anti-commuting:

$$x^i \cdot x^j = x^j \cdot x^i \,, \quad x^i \cdot \xi^a = \xi^a \cdot x^i \,, \quad \xi^a \cdot \xi^b = -\xi^b \cdot \xi^a \,.$$

- Superroughly speaking, a $(p|q)$-dimensional supermanifold M is a topological space that is locally isomorphic to $\mathbb{R}^{p|q}$. There is an associated p-dimensional manifold $|M|$ called the body of M.
- In this talk, I will be interested only in local coordinates, so we can just think $M = \mathbb{R}^{p|q}$ and $|M| = \mathbb{R}^p$.
- Smooth functions on M are polynomials on the anticommuting variables ξ with functions on the commuting variables x as coefficients, e.g. in $\mathbb{R}^{p|2}$ these are of the form

$$f(x^1, \ldots x^p, \xi^1, \xi^2) = f_0(x) + f_1(x)\,\xi^1 + f_2(x)\,\xi^2 + f_{12}(x)\,\xi^1 \cdot \xi^2 \,.$$

A supercrash course on supergeometry

- The fact that there are commuting and anti-commuting coordinates makes supermanifolds equipped with a $\mathbb{Z}_2$-grading.
- We call objects with $\mathbb{Z}_2$-degree 0 (resp. 1) **even** (resp. **odd**).
- A tangent vector v at a point p is defined as a superderivation on the space of functions on M:

$$v(f \cdot g) = v(f) \cdot g(p) + (-1)^{|v|\,|f|} f(p) \cdot v(g)\,,$$

  where $|\cdot|$ denotes the $\mathbb{Z}_2$-grading.
- Coordinates (x^i, ξ^a) induce a basis $(\partial_{x^i}, \partial_{\xi^a})$ of the tangent space T_pM at p such that

$$\partial_{x^i}(x^j) = \delta_i^j\,, \quad \partial_{\xi^a}(\xi^b) = \delta_a^b\,, \quad \partial_{x^i}(\xi^a) = 0 = \partial_{\xi^a}(x^i)\,.$$

- With this, it is possible to extend the notions of vector field, differential form, (co)tangent bundle, and (co)distribution.

A supercrash course on supergeometry

- The wedge product now depends on the $\mathbb{Z}_2$-grading, as well as the usual $\mathbb{N}$-grading:

$$\alpha \wedge \beta = (-1)^{kl+|\alpha|\,|\beta|}\beta \wedge \alpha\,,$$

for any k-form α and any l-form β.

Frobenius theorem

Definition

A distribution $D \subseteq \mathrm{T}M$ (resp. codistribution $D \subseteq \mathrm{T}^*M$) on a homogeneity (super)manifold (M, ∇) is called a **homogeneous distribution** if the tangent lift $\mathrm{d}_\mathrm{T}\nabla$ (resp. the cotangent lift $\mathrm{d}_\mathrm{T}^*\nabla$) is tangent to D.

Theorem (Grabowska and Grabowski, 2024)

$D \subseteq \mathrm{T}M$ is a homogeneous distribution iff it is locally generated by homogeneous vector fields.

Corollary

$D \subseteq \mathrm{T}^*M$ is a homogeneous codistribution iff it is locally generated by homogeneous one-forms.

Homogeneous Frobenius theorem

Theorem (Grabowski and L. G.)

Let (M, ∇) be a homogeneity (super)manifold, and let D be a homogeneous <u>involutive</u> distribution of rank $r \mid s$ on M. Around every point $x_0 \in |M|$ at which ∇ vanishes, there exists a system of homogeneous coordinates (t^i, θ^j) such that D is locally generated by $\partial_{t^1}, \ldots, \partial_{t^r}, \partial_{\theta^1}, \ldots, \partial_{\theta^s}$.

Remark

If $f \in \mathscr{C}^\infty(M)$ is a w-homogeneous function on a homogeneity manifold (M, ∇) with non-zero weight w, then it vanishes at every point $x_0 \in M$ at which ∇ vanishes. Indeed,

$$f(x_0) = \frac{1}{w} \left(\mathcal{L}_\nabla f\right)(x_0) = \frac{1}{w} \left(\iota_\nabla \mathrm{d} f\right)(x_0) = 0 \,.$$

In other words, a homogeneous function non-vanishing at x_0 is necessarily of degree zero.

Consequently, if a homogeneous one-form θ does not vanish at x_0, then $\deg(\theta) \in \Gamma$, with $\Gamma \subset \mathbb{R}$ the set of weights of any system of homogeneous coordinates (x^a) around x_0. Otherwise, all the coefficients of θ in the basis $(\mathrm{d}x^a)$ would vanish at x_0.

Lemma

Let D be an involutive homogeneous distribution on a homogeneity (super)manifold (M, ∇). Then, on a neighbourhood of each $x_0 \in |M|$ at which ∇ vanishes, there exist linearly independent (super)commuting homogeneous vector fields locally generating D.

Lemma

Let X be an even $(-\lambda)$-homogeneous even vector field on a homogeneity supermanifold (M, ∇) (with $\lambda \in \mathbb{R}$). Assume that, at a point $x_0 \in |M|$,

$$\nabla(x_0) = 0\,, \quad X(x_0) \neq 0\,.$$

Then, in a neighbourhood of x_0, there exists a system of homogeneous coordinates (y^a, ξ^α) such that $X = \partial_{y^1}$.

Lemma

Let χ be an odd $(-\lambda)$-homogeneous vector field on a homogeneity supermanifold (M, ∇) (with $\lambda \in \mathbb{R}$). If $\chi^2 = 0$ and χ generates a distribution, then, in a neighbourhood of each $x_0 \in |M|$ at which ∇ vanishes, there exist a system of homogeneous local coordinates (t^i, θ^I) such that $\chi = \partial_{\theta^1}$.

Darboux theorems

Homogeneous Poincaré lemma

Lemma (Grabowska and Grabowski, 2024)

Let ω be a λ-homogeneous k-form (with $k > 0$ and $\lambda \in \mathbb{R}$) on a homogeneity (super)manifold (M, ∇). In a neighbourhood of each $x_0 \in M$ such that $\nabla(x_0) = 0$, there exists a $(k-1)$-form α such that:

1. $\mathrm{d}\alpha = \omega$,
2. *α is λ-homogeneous,*
3. $\alpha(x_0) = 0$.

Homogeneous symplectic Darboux theorem

Theorem (Grabowska and Grabowski, 2024)

Let ω be (λ, σ)-homogeneous symplectic form on a homogeneity (super)manifold (M, ∇) (with $\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{Z}_2$). Around every $x_0 \in |M|$ such that $\nabla(x_0) = 0$, there is a system of homogeneous coordinates (q^i, p_i, ξ^I) such that

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i + \sum_I \varepsilon^I \mathrm{d}\xi^I \wedge \mathrm{d}\xi^I , \quad \varepsilon^I = \pm 1 .$$

Definition

A **presymplectic form** ω on a (super)manifold M is a closed 2-form of constant rank r. Its **characteristic distribution** $C_\omega \subseteq TM$ is given by

$$C_\omega = \ker \omega \,.$$

Proposition

The characteristic distribution C_ω is an integrable distribution. Moreover, if ω is homogeneous (w.r.t. a weight vector field ∇ on M), then C_ω is a homogeneous distribution.

Homogeneous presymplectic Darboux theorem

Theorem (Grabowski and L. G.)

Let ω be (λ, σ)-homogeneous presymplectic form on a homogeneity (super)manifold (M, ∇) (with $\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{Z}_2$). Around every $x_0 \in |M|$ such that $\nabla(x_0) = 0$, there is a system of homogeneous coordinates $(q^i, p_i, \xi^I, z^a, \chi^b)$ such that

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i + \sum_I \varepsilon^I \mathrm{d}\xi^I \wedge \mathrm{d}\xi^I, \quad \varepsilon^I = \pm 1 .$$

Homogeneous presymplectic Darboux theorem

Proof.

Since C_ω is integrable, there exists a homogeneous chart $(U; y^1, \dots y^n)$ around x_0 such that

$$C_\omega = \langle \partial_{y^1}, \dots, \partial_{y^{n-2r}} \rangle ,$$

and $(y^{n-2r+1}, \dots, y^n)$ are coordinates on the space of leaves $\mathfrak{F}$ of the integral foliation. This space is endowed with a weight vector field

$$\nabla_{\mathfrak{F}} = \pi_* \nabla = \sum_{i=n-2r+1}^{n} w_i \cdot y^i \partial_{\tilde{y}^i} ,$$

and a symplectic form $\tilde{\omega}$ such that $\omega = \pi^* \tilde{\omega}$. In fact, $\tilde{\omega}$ is λ-homogeneous w.r.t. $\nabla_{\mathfrak{F}}$. Applying the Homogeneous Darboux theorem to $\tilde{\omega}$ and lifting the resulting coordinates to M, the result follows. □

Class of a one-form

Definition

Let α be a k-form on a supermanifold M. The subset

$$\chi(\alpha) = \ker(\alpha) \cap \ker(\mathrm{d}\alpha) \subseteq \mathrm{T}M$$

is called the **characteristic set** of α.
If $\chi(\alpha)$ is a distribution, it is called the **characteristic distribution** of α, we say that α is **regular**, and the corank of $\chi(\alpha)$ as a sub-bundle of $\mathrm{T}M$ is called the **class of** α:

$$c_\alpha := \operatorname{corank}\left(\chi(\alpha)\right) .$$

Class of a one-form

Proposition

If α is a regular form, then $\chi(\alpha)$ is involutive and α is $\chi(\alpha)$-invariant.

Proof.

For any pair of sections X and Y of $\chi(\alpha) = \ker \alpha \cap \ker \mathrm{d}\alpha$,

$$\iota_{[X,Y]}\alpha = [\iota_X, \mathcal{L}_Y]\alpha = \iota_X \iota_Y \mathrm{d}\alpha = 0\,, \quad \iota_{[X,Y]}\mathrm{d}\alpha = 0\,,$$

and

$$\mathcal{L}_X \alpha = \mathrm{d}(\iota_X \alpha) + \iota_X \mathrm{d}\alpha = 0\,.$$

$\square$

Non-degenerate one-forms

Definition

A regular one-form α on a (super)manifold M is called **non-degenerate** if its characteristic foliation is trivial:

$$\chi(\alpha) = \{0_M\}\,,$$

or equivalently,

$$c_\alpha = \dim(M)\,.$$

Non-degenerate one-forms

The annihilator of $\chi(\alpha)$ is given by

$$\left(\chi(\alpha)\right)^\circ = (\ker\alpha \cap \ker \mathrm{d}\alpha)^\circ = (\ker\alpha)^\circ + (\ker \mathrm{d}\alpha)^\circ = \langle\alpha\rangle + \operatorname{Im}(\flat_{\mathrm{d}\alpha}),$$

where $\flat_{\mathrm{d}\alpha} \colon \mathrm{T}M \ni v \mapsto \iota_v \mathrm{d}\alpha \in \mathrm{T}^*M$.

The form is non-degenerate iff $\left(\chi(\alpha)\right)^\circ = \mathrm{T}^*M$, so there are two possible cases for $\dim M \geq 2$:

1. $\mathrm{T}^*M = \langle\alpha\rangle \oplus \operatorname{Im}(\flat_{\mathrm{d}\alpha}) \Longrightarrow c_\alpha = \operatorname{rank}(\flat_{\mathrm{d}\alpha}) + 1$ (**contact form**),
2. $\mathrm{T}^*M = \operatorname{Im}(\flat_{\mathrm{d}\alpha}) \Longrightarrow c_\alpha = \operatorname{rank}(\flat_{\mathrm{d}\alpha})$ (**symplectic potential**).

Non-degenerate one-forms

The situation $\dim M = 1$, on the other hand, is trivial, since then every one-form is closed.

Remark

The rank of an even 2-form (or a standard 2-form on a classical manifold) is always even, so a non-degenerate 2-form is a contact form (resp. a symplectic potential) iff $\dim M$ is odd (resp. even).

Since $\chi(\alpha)$ is an involutive homogeneous distribution, the homogeneous Frobenius theorem provides a system of homogeneous coordinates $(t^1, \ldots t^p, \theta^1, \ldots, \theta^q)$ such that the projection $\pi\colon M \to M/\chi(\alpha)$ onto the space of leaves reads

$$\pi(t^1, \ldots t^p, \theta^1, \ldots, \theta^q) = (t^{r+1}, \ldots, t^p, \theta^{s+1}, \ldots, \theta^q)\,.$$

Furthermore, α induces a non-degenerate one-form α_{red} on $M/\chi(\alpha)$ uniquely determined by

$$\pi^*(\alpha_{\mathrm{red}}) = \alpha\,.$$

Remark

Of course, the foliation $\pi\colon M \to M/\chi(\alpha)$ may not be simple (i.e., $M/\chi(\alpha)$ may not have a smooth structure making π a surjective submersion). However, since we are interested in a local characterisation (Darboux coordinates), it suffices to restrict to charts.

Definition

Let α be a regular one-form on a (super)manifold M. We call α a **precontact form** (resp. a **presymplectic** potential) if the reduced one-form α_{red} on $M/\chi(\alpha)$ is a **contact form** (resp. a **symplectic** potential).

Darboux theorem for homogeneous one-forms

Theorem (Grabowski and L. G.)

Let α be a regular λ-homogeneous one-form on a homogeneity supermanifold (M, ∇). Around each $x_0 \in M$, there exists a system of homogeneous coordinates in which α has a canonical expression:

1. $\alpha = \mathrm{d}z - p_a \mathrm{d}x^a + \frac{\varepsilon_j}{2}\theta^j \mathrm{d}\theta^j$, *if α is even and precontact,*
2. $\alpha = \mathrm{d}\xi - \theta^a \mathrm{d}x^a$, *if α is odd and precontact,*
3. $\alpha = p_a \mathrm{d}x^a + \frac{\varepsilon_j}{2}\theta^j \mathrm{d}\theta^j$, *if α is even and a presymplectic potential,*
4. $\alpha = \theta^a \mathrm{d}x^a$, *if α is odd and a presymplectic potential.*

Here $\varepsilon_j = \pm 1$.

Darboux theorem for homogeneous one-forms

Corollary

Let α be a regular λ-homogeneous one-form on a homogeneity manifold (M, ∇). Around each $x_0 \in M$, there exists a system of homogeneous coordinates in which α has a canonical expression:

1. $\alpha = \mathrm{d}z - p_a \mathrm{d}x^a$, *if α is precontact,*
2. $\alpha = p_a \mathrm{d}x^a$, *if α is a presymplectic potential.*

Darboux theorem for homogeneous one-forms

Proof for even contact forms:

If α is a contact form, there is a unique Reeb vector field R on M such that

$$\iota_R \alpha = 1\,, \iota_R \mathrm{d}\alpha = 0\,.$$

It is $-\lambda$-homogeneous and nowhere-vanishing. Therefore, we can apply the homogeneous straightening lemma and write $R = \partial_z$ in some system of homogeneous coordinates.

On the other hand, the 2-form $\mathrm{d}\alpha$ is independent of z. In fact, $\mathrm{d}\alpha|_{\{z=0\}}$ is homogeneous symplectic, so it can be written in the canonical form $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i + \sum_I \varepsilon^I \mathrm{d}\xi^I \wedge \mathrm{d}\xi^I$.

Finally, applying the Homogeneous Poincaré Lemma to the closed 1-form $\alpha - p_a \mathrm{d}x^a + \frac{\varepsilon_j}{2}\theta^j \mathrm{d}\theta^j$, we obtain the canonical form for α.

Darboux theorem for homogeneous one-forms

Proof for even symplectic potentials:

If α is an even presymplectic potential on M, then we can define the even contact form

$$\eta = \mathrm{d}t + \alpha\,,$$

on $M \times \mathbb{R}$, which is λ-homogeneous w.r.t. the weight vector field

$$\overline{\nabla} = \nabla + \lambda t \partial_t\,,$$

where t is the canonical coordinate on $\mathbb{R}$. There exists a system of homogeneous coordinates (q^a, p_a, z, θ^j) around $(x_0, 0)$ in which it reads

$$\eta = \mathrm{d}z - p_a \mathrm{d}x^a + \frac{\varepsilon_j}{2}\theta^j \mathrm{d}\theta^j, \quad \varepsilon_j = \pm 1\,.$$

We can easily check that $z = t$; while the coordinates (q^a, p_a, θ^j) project into coordinates on M, which are canonical coordinates for θ.

Darboux theorem for homogeneous one-forms

The proofs for odd contact forms and odd presymplectic potentials are similar.

Proof for degenerate forms:
The canonical form of a degenerate α can be obtained by pullbacking the canonical form of α_{red}, which is non-degenerate, i.e., either a contact form or a symplectic potential.

$\square$

Example: vector bundles and linear one-forms

- Recall that a VB $\pi\colon E \to B$ is endowed with the Euler vector field

$$\nabla_E = \sum_{i=1}^{k} y^i \partial_{y^i} .$$

  In this case, the set of weights of the weight vector field is simply $\Gamma = \{0, 1\}$.
- Hence, any non-vanishing homogeneous one-form θ on E is either basic (if $\deg(\theta) = 0$) or linear (if $\deg(\theta) = 1$).
- Notice that the possible classes of one-forms are restricted by the dimensions of the base space (n) and the fibers (k):

- Presymplectic potentials have class $2s+2$ with $1 \leq s+1 \leq \min\{k, n\}$. Indeed, for a one-form θ of such class, and around any point on the zero-section of E, there exists a system of homogeneous coordinates (q^b, p_b, z^ν) such that

$$\theta = \sum_{b=0}^{s} p_b \mathrm{d} q^b \,,$$

  with

$$\deg(q^b), \deg(p_b), \deg(z^\nu) \in \{0, 1\} \quad \text{and} \quad \deg(q^b) + \deg(p_b) = 1 \,.$$

  In particular, at least $s+1$ of the coordinates have degree 0, and at least other $s+1$ of them have degree 1.

- Precontact forms have class $2s + 1$ with $1 \leq s + 1 \leq \min\{k, n + 1\}$. Indeed, for a one-form θ of such class, and around any point on the zero-section of E, there exists a system of homogeneous coordinates (q^j, p_j, z, t^c) such that

$$\theta = \mathrm{d}z + \sum_{j=1}^{s} p_j \mathrm{d}q^j \,,$$

with

$$\deg(q^j), \deg(p_j), \deg(t^c) \in \{0, 1\} \,, \quad \deg(q^j) + \deg(p_j) = 1 \quad \text{and} \quad \deg(z) =$$

In particular, at least s of the coordinates have degree 0, and at least $s + 1$ of them have degree 1.

- Systems of bundle coordinates on $\pi \colon E \to B$ are just systems of homogeneous coordinates on (E, ∇_E) and *vice versa*. This means that any system of homogeneous coordinates $(\tilde{x}^a, \tilde{y}^i)$ on (E, ∇_E), with $\deg(\tilde{x}^a) = 0$ and $\deg(\tilde{y}^i) = 1$, induces a system of coordinates $(\bar{x}^a)$ on the base space B such that $\tilde{x}^a = \pi^* \bar{x}^a$.
- In this way, given a linear one-form of constant class on a VB, our results allow constructing coordinates which are simultaneously Darboux for the one-form and adapted to the VB structure.

Conclusions

Summary

We have obtained homogeneous local coordinates (around the zeros of ∇) which are canonical for homogeneous objects:

- homogeneous involutive distributions,
- homogeneous (pre)symplectic forms,
- homogeneous one-forms.

Future research

- Homogeneous multisymplectic forms
- Homogeneous Poisson structures and homogeneous Weinstein splitting coordinates
- Multiple gradings: ∇_1 and ∇_2 such that $[\nabla_1, \nabla_2] = 0$

Main references

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Thank you for your attention!

✉ Feel free to contact me at alopez-gordon@impan.pl

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