Forced Hamiltonian and Lagrangian systems

Symmetries, reduction and Hamilton-Jacobi theory

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Symplectic structure on TQ induced by the Lagrangian I

- A **symplectic manifold** (M, ω) is an $2m$-dimensional manifold M endowed with a symplectic form ω (i.e., a closed and non-degenerate 2-form).
- The **vertical endomorphism** $S : T(TQ) \to T(TQ)$ is given by

$$S\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial \dot{q}^i}, \qquad S\left(\frac{\partial}{\partial \dot{q}^i}\right) = 0.$$

- Its adjoint $S^* : T^*(TQ) \to T^*(TQ)$ is given by

$$S^*(\mathrm{d}q^i) = 0, \qquad S^*(\mathrm{d}\dot{q}^i) = \mathrm{d}q^i.$$

Symplectic structure on TQ induced by the Lagrangian II

- Consider a Lagrangian function L on TQ.
- The Poincaré-Cartan forms are given by

$$\theta_L = S^*(\mathrm{d}L), \qquad \omega_L = -\mathrm{d}\theta_L.$$

- Hereinafter, L will be assumed to be regular, i.e., ω_L is symplectic.
- The Liouville vector field Δ on TQ is given by

$$\Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}.$$

SODE

- A **second order differential equation** (**SODE**) is a vector field ξ on TQ that is a section of both $\tau_{TQ} : TTQ \to TQ$ and $T\tau_Q : TTQ \to TQ$.
- Locally,

$$\xi = \dot{q}^i \frac{\partial}{\partial q^i} + \xi^i(q^i, \dot{q}^i) \frac{\partial}{\partial \dot{q}^i}.$$

- Clearly, ξ is a SODE if and only if

$$S(\xi) = \Delta.$$

- A **solution** of a SODE ξ is a curve $\sigma(t) = (q^i(t))$ on Q such that its canonical lift to TQ is an integral curve of ξ, given by

$$\frac{\mathrm{d}^2 q^i}{\mathrm{d}t^2} = \xi^i \left(q^i, \frac{\mathrm{d}q^i}{\mathrm{d}t} \right), \quad 1 \leq i \leq n = \dim Q.$$

Forced Euler-Lagrange equations

- An external force is represented by a semibasic 1-form α on TQ. Locally,
$$\alpha = \alpha_i(q, \dot{q}) \, \mathrm{d}q^i .$$
- The dynamics is determined by the **forced Euler-Lagrange vector field** $\xi_{L,\alpha}$, given by
$$\iota_{\xi_{L,\alpha}} \omega_L = \mathrm{d}E_L + \alpha,$$
where $E_L = \Delta(L) - L$.
- $\xi_{L,\alpha}$ is a SODE, with solutions given by
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = -\alpha_i, \quad 1 \leq i \leq n.$$

Vertical and complete lifts of a vector field

- Consider a vector field X on Q locally given by

$$X = X^i \frac{\partial}{\partial q^i}.$$

- Its **vertical lift** is the vector field X^v on TQ given by

$$X^v = X^i \frac{\partial}{\partial \dot{q}^i}.$$

- Its **complete lift** is the vector field X^c on TQ given by

$$X^c = X^i \frac{\partial}{\partial q^i} + \dot{q}^j \frac{\partial X^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i}.$$

Rayleigh forces

- An **Rayleigh force** is an external force of the form

$$\bar{R} = S^*(\mathrm{d}\mathcal{R}),$$

where $\mathcal{R} : TQ \to \mathbb{R}$ is the **Rayleigh potential** or **Rayleigh dissipation function**.

- $\mathcal{R}$ expresses the energy dissipated away by the system:

$$\frac{\mathrm{d}}{\mathrm{d}t} E_L \circ \sigma(t) = -\Delta(\mathcal{R}) \circ \sigma(t),$$

with σ an integral curve of $\xi_{L,\bar{R}}$.

Dissipative bracket

- The **dissipative bracket** of a pair of functions f and g on (TQ, ω_L) is given by

$$[f, g] := (SX_f)(g) = \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)^{-1} \frac{\partial f}{\partial \dot{q}^i} \frac{\partial g}{\partial \dot{q}^j}$$

- It is bilinear and symmetric
- It satisfies the Leibniz rule:

$$[fg, h] = [f, h]g + f[g, h]$$

- f is a constant of the motion of $(L, \mathcal{R})$ iff

$$\{f, E_L\} - [f, \mathcal{R}] = 0.$$

Noether theorem

Theorem (Noether's theorem for forced Lagrangian systems)

Let X be a vector field on Q. Then $X^c(L) = \alpha(X^c)$ if and only if $X^v(L)$ is a constant of the motion.

- A vector field X on Q satisfying these conditions is called a **symmetry of the forced Lagrangian** (L, α).
- For a Rayleigh system $(L, \mathcal{R})$, this is equivalent to

$$X^c(L) = X^v(\mathcal{R}).$$

Example (Fluid resistance)

- Consider a body of mass m moving along 1 dimension through a fluid that fully encloses it.
- The Rayleigh potential associated to the drag force is

$$\mathcal{R} = \frac{k}{3}\dot{q}^3, \qquad k = \frac{1}{2}CA\rho; \qquad L = \frac{1}{2}m\dot{q}^2.$$

- Consider the vector field

$$X = e^{kq/m}\frac{\partial}{\partial q}.$$

- $X^c(L) = X^v(\mathcal{R}) \Longrightarrow X^v(L) = me^{kq/m}\dot{q}$ is a constant of the motion.
- When $k \to 0$, X is the generator of translations and the conservation of momentum is recovered.

Other point-like symmetries I

- A **Lie symmetry** is a vector field X on Q such that

$$[X^c, \xi_{L,\alpha}] = \mathcal{L}_{X^c}\xi_{L,\alpha} = 0$$

- If $\mathcal{L}_{X^c}\alpha_L$ is closed, then X is a Lie symmetry if and only if

$$\mathcal{L}_{X^c}\alpha = -\mathrm{d}(X^c(E_L)).$$

- A **Noether symmetry** is a vector field X on Q such that

$$\mathcal{L}_{X^c}\alpha_L = \mathrm{d}f, \qquad X^c(E_L) + \alpha(X^c) = 0.$$

- If $\mathcal{L}_{X^c}\alpha_L = \mathrm{d}f$, then X is a Noether symmetry if and only if $f - X^v(L)$ is a conserved quantity.

Other point-like symmetries II

- For a Rayleigh system $(L, \mathcal{R})$, if $\mathcal{L}_{X^c}\alpha_L = \mathrm{d}f$, then X is a Noether symmetry if and only if

$$X^c(E_L) + X^v(\mathcal{R}) = 0.$$

- If X is a Noether symmetry, it is also a symmetry of the forced Lagrangian if and only if $\mathcal{L}_{X^c}\alpha_L = 0$.
- If X is a Noether symmetry, it is also a Lie symmetry if and only if

$$\iota_{X^c}\mathrm{d}\alpha = 0.$$

Non-point-like symmetries I

- A vector field $\tilde{X}$ on TQ is called a **dynamical symmetry** if

$$[\tilde{X}, \xi_{L,\alpha}] = 0.$$

- A vector field $\tilde{X}$ on TQ is called a **Cartan symmetry** if

$$\mathcal{L}_{\tilde{X}} \alpha_L = \mathrm{d} f, \qquad \tilde{X}(E_L) + \alpha(\tilde{X}) = 0$$

- X is a Lie symmetry if and only if X^c is a dynamical symmetry.
- X is a Noether symmetry if and only if X^c is a Cartan symmetry.

Non-point-like symmetries II

- If $\mathcal{L}_{\tilde{X}}\alpha_L$ is closed, then $\tilde{X}$ is a dynamical symmetry if and only if

$$\mathrm{d}(\tilde{X}(E_L)) = -\mathcal{L}_{\tilde{X}}\alpha.$$

- A Cartan symmetry is a dynamical symmetry if and only if

$$\iota_{\tilde{X}}\mathrm{d}\alpha = 0.$$

- If $\mathcal{L}_{\tilde{X}}\alpha_L = \mathrm{d}f$, then $\tilde{X}$ is a Cartan symmetry if and only if $f - (S\tilde{X})(L)$ is a constant of the motion.
- For a Rayleigh system $(L, \mathcal{R})$, $\tilde{X}$ is a Cartan symmetry if and only if

$$\tilde{X}(E_L) + (S\tilde{X})(\mathcal{R}) = 0.$$

Momentum map

- Consider a G-invariant regular Lagrangian L on TQ, where G is a Lie group with Lie algebra $\mathfrak{g}$ and dual Lie algebra $\mathfrak{g}^*$.
- Assume the G-action to be free and proper.
- The **natural momentum map** is given by

$$J : TQ \to \mathfrak{g}^*$$
$$\langle J(x), \xi \rangle = \theta_L(\xi_Q^c)$$

  for each $\xi \in \mathfrak{g}$.
- For each $\xi \in \mathfrak{g}$, we can introduce a function on TQ:

$$J^\xi : TQ \to \mathbb{R}$$
$$x \mapsto \langle J(x), \xi \rangle$$

Lemma

Consider a forced Lagrangian system (L, α). *Let* $\xi \in \mathfrak{g}$. *Then*

1. J^ξ *is a conserved quantity if and only if*

$$\alpha(\xi_Q^c) = 0.$$

2. *If the previous equation holds, then* ξ *leaves* α *invariant if and only if*

$$\iota_{\xi_Q^c} \mathrm{d}\alpha = 0.$$

In addition, the vector subspace of $\mathfrak{g}$ *given by*

$$\mathfrak{g}_\alpha = \left\{ \xi \in \mathfrak{g} \mid \alpha(\xi_Q^c) = 0,\ \iota_{\xi_Q^c} \mathrm{d}\alpha = 0 \right\}$$

is a Lie subalgebra of $\mathfrak{g}$.

Level sets

- J^{ξ} is a constant of the motion $\forall \xi \in \mathfrak{g}_\alpha \Rightarrow J_\alpha^{-1}(\mu)$ is left invariant by the flow of $\xi_{L,\alpha}$.
- Therefore the integral curves of $\xi_{L,\alpha}$ are contained in level sets $J_\alpha^{-1}(\mu) \subset TQ$.
- In other words, a trajectory starting on a particular level set $J_\alpha^{-1}(\mu)$ will not go out of the level set along the motion.

Adjoint and coadjoint actions and isotropy group

For $g \in G_\alpha,\ \xi \in \mathfrak{g}_\alpha,\ \mu \in \mathfrak{g}_\alpha^*$:

- The adjoint action is given by

$$\operatorname{Ad}_g \xi = \left.\frac{\mathrm{d}}{\mathrm{d}t} g \exp(t\xi) g^{-1}\right|_{t=0}.$$

- The coadjoint action is given by

$$\left\langle \operatorname{Ad}_g^* \mu, \xi \right\rangle = \langle \mu, \operatorname{Ad}_g \xi \rangle$$

- The isotropy group for $\mu \in \mathfrak{g}_\alpha^*$ is

$$(G_\alpha)_\mu = \left\{ g \in G_\alpha \mid \operatorname{Ad}_g^* \mu = \mu \right\}$$

Theorem

Consider a $\mathfrak{g}_\alpha$-invariant forced Lagrangian system (L, α) on TQ. Let $\mu \in \mathfrak{g}_\alpha^*$. Then:

1. *The quotient space $(TQ)_\mu := J_\alpha^{-1}(\mu)/(G_\alpha)_\mu$ is endowed with an induced symplectic structure ω_μ, given by*

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega_L,$$

   where $\pi_\mu : J_\alpha^{-1}(\mu) \to (TQ)_\mu$ and $i_\mu : J_\alpha^{-1}(\mu) \hookrightarrow TQ$.

2. *The reduced Lagrangian L_μ is given by*

$$L_\mu \circ \pi_\mu = L \circ i_\mu.$$

3. *The reduced external force α_μ is given by*

$$\pi_\mu^* \alpha_\mu = i_\mu^* \alpha.$$

Standard Hamilton-Jacobi problem

- The Hamilton-Jacobi problem consists in finding a characteristic function W on Q such that

$$H\left(q^i, \frac{\partial W}{\partial q^i}\right) = E.$$

- Geometrically, this equation can be written as

$$\gamma^* H = E,$$

with $\gamma = \mathrm{d}W$ a section of T^*Q.

Hamilton-Jacobi problem for (H, β)

Theorem

Let γ be a closed 1-form on Q. Then the following conditions are equivalent:

1. $\mathrm{d}(H \circ \gamma) = -\gamma^* \beta$,
2. *for every curve $\sigma : \mathbb{R} \to Q$ such that*

$$\dot{\sigma}(t) = T\pi_Q \circ X_{H,\beta} \circ \gamma \circ \sigma(t)$$

   for all t, then $\gamma \circ \sigma$ is an integral curve of $X_{H,\beta}$;

3. $\operatorname{Im} \gamma$ *is a Lagrangian submanifold of T^*Q and $X_{H,\beta}$ is tangent to it.*

If γ satisfies these conditions, it is called a solution of the Hamilton-Jacobi problem for (H, β),

Complete solutions I

- A map $\Phi : Q \times \mathbb{R}^n \to T^*Q$ is called **complete solution of the Hamilton-Jacobi problem** for (H, β) if
  1. Φ is a local diffeomorphism,
  2. for any $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, the map

$$\begin{aligned} \Phi_\lambda : Q &\to T^*Q \\ q &\mapsto \Phi_\lambda(q) = \Phi(q, \lambda_1, \dots, \lambda_n) \end{aligned}$$

  is a solution of the Hamilton-Jacobi problem for (H, β).
- Assume Φ to be a global diffeomorphism.

Complete solutions II

- Consider the functions given by

$$f_a = \pi_a \circ \Phi^{-1} : T^*Q \to \mathbb{R},$$

where π_a denotes the projection over the a-th component of $\mathbb{R}^n$.

- The functions f_a are constants of the motion. Moreover, they are in involution, i.e.,

$$\{f_a, f_b\} = 0$$

Example

Consider a n-dimensional forced Hamiltonian system (H, β), with

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2, \qquad \beta = \sum_{i=1}^{n} \kappa_i p_i^2 \mathrm{d}q_i.$$

The functions

$$f_a = e^{\kappa_a q^a} p_a, \qquad a = 1, \ldots, n.$$

are constants of the motion in involution. The 1-form γ on Q given by

$$\gamma = \sum_{i=1}^{n} \lambda_i e^{-\kappa_i q^i} \mathrm{d}q^i$$

is a complete solution of the Hamilton-Jacobi problem.

Reduction and reconstruction of the Hamilton-Jacobi problem

- Let (H, β) be a forced Hamiltonian system on T^*Q.
- Let G be a Lie group that acts freely and properly on T^*Q. Suppose that this action preserves θ_Q, H and β.
- Then, we can introduce a reduced Hamiltonian $\tilde{H}$ and a reduced external force $\tilde{\beta}$ on $T^*(Q/G)$.
- If γ is a G-invariant solution of the Hamilton-Jacobi problem for H, β, then it induces a solution $\tilde{\gamma}$ of the Hamilton-Jacobi problem for $(\tilde{H}, \tilde{\beta})$.
- Conversely, we can reconstruct γ from $\tilde{\gamma}$.

Example (Calogero-Moser system with a linear Rayleigh force)

- Consider a forced Hamiltonian system (H, R) on $T^*\mathbb{R}^2$, where

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + \frac{1}{(x-y)^2}\right), \qquad \tilde{R} = (p_x + p_y)(\mathrm{d}x - \mathrm{d}y).$$

- Consider the action $\Phi\left(t, (x, y)\right) = (t + x, t + y)$ of $\mathbb{R}$ on $\mathbb{R}^2$.
- Clearly, $(H, \tilde{R})$ is invariant under Φ^{T^*}. The momentum map is $J(x, y, p_x, p_y) = p_x + p_y$.
- We can identify $J^{-1}(\mu)/\mathbb{R}$ with $\mathbb{R}^2$, with coordinates (q, p) and the natural projection $\pi : (x, y, p, \mu - p) \mapsto (x - y, p)$.
- $\tilde{\gamma}_\lambda = \mathrm{d}\tilde{S}_\lambda \rightsquigarrow \gamma_\lambda = \mathrm{d}S_\lambda$, where the generating functions are

$$\tilde{S}_\lambda(q) = \frac{1}{2}q^2 - \frac{1}{2\mu q} + \lambda q, \qquad S_\lambda(x, y) = \tilde{S}_\lambda(x - y) + \mu y.$$

Main references

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Thank you!