Homogeneous Darboux theorems

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The interest of gradings

There are several scenarios in geometry and physics in which a $(\mathbb{N}, \mathbb{Z}, \mathbb{Z}_2, \mathbb{R} \ldots)$ grading appears:

- the algebra of exterior forms with the exterior product $\left(\Omega^\bullet(M), \wedge\right)$,
- the spin of particles,
- intensive/extensive variables in thermodynamics,
- symplectisation/Poissonisation of contact/Jacobi manifolds,
- supermanifolds,
- higher tangent bundles.

Theorem (Euler)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, and k an integer. The following assertions are equivalent:

1. $f(t \cdot x) = t^k f(x)\,, \quad \forall\, t \in \mathbb{R} \setminus \{0\}\,,\ \forall\, x \in \mathbb{R}^n\,,$
2. *f is an eigenfunction of $X = \sum_{i=1}^{n} x^i \partial_{x^i}$ with eigenvalue k, namely*

$$X(f) = k \cdot f\,.$$

Definition

A function f satisfying any of the equivalent conditions above is called **homogeneous of degree** k or k**-homogeneous**.

We can extend this notion to a manifold M^n by considering a vector field $X \in \mathfrak{X}(M)$ which is locally of the form

$$X = \sum_{i=1}^{n} x^i \partial_{x^i}$$

in a certain atlas.

Definition

An (even) vector field ∇ on a (super)manifold M is called a **weight vector field** if in a neighbourhood of every point of (the body of) M there are coordinates (x^a) such that

$$\nabla = \sum_{a=1}^{n} w_a \cdot x^a \partial_{x^a}, \quad w_a \in \mathbb{R}.$$

Such coordinates are called **homogeneous coordinates**, and the pair (M, ∇) is called a **homogeneity (super)manifold**.

Definition

Let (M, ∇) be a homogeneity (super)manifold and $w \in \mathbb{R}$. A tensor field A on M is called **homogeneous of degree** w or w-**homogeneous** if

$$\mathcal{L}_{\nabla} A = w \cdot A\,.$$

Example (Trivial)

The zero-section of the tangent bundle makes any (super)manifold a homogeneity (super)manifold:

$$\nabla \equiv 0 \,.$$

This means that all the subsequent results I shall present still hold if you forget the adjective "homogeneous".

Example (Vector bundles)

Let $\pi \colon E \to M$ be a vector bundle (VB). The Euler vector field $\nabla_E \in \mathfrak{X}(E)$, i.e. the infinitesimal generator of homotheties on the fibers, is a weight vector field. In bundle coordinates,

$$\pi \colon (x^i, y^a) \mapsto (x^i), \quad \nabla_E = \sum_a y^a \partial_{y^a} .$$

Remark

The structure of VB on E is uniquely determined[a] by its structure of manifold and a smooth action of the monoid $(\mathbb{R}, \cdot)$ generated by ∇_E

[a] See Grabowski and Rotkiewicz, *J. Geom. Phys.* **59** (2009).

Example (Exact symplectic manifolds)

Let (M, ω) be a symplectic manifold. Then, the following statements are equivalent:

1. ω is exact, i.e. there exists a $\theta \in \Omega^1(M)$ such that $\omega = \mathrm{d}\theta$,
2. there exists a **Liouville vector field** $\nabla \in \mathfrak{X}(M)$ such that $\mathcal{L}_\nabla \omega = \omega$.

In fact, since $\mathfrak{X}(M) \ni X \mapsto \iota_X \omega \in \Omega^1(M)$ is an isomorphism, given θ (resp. ∇), we can univocally define θ (resp. ∇) by the relation

$$\iota_\nabla \omega = \theta\,.$$

The Liouville vector field is a weight vector field. Indeed, in Darboux coordinates (q^i, p_i) for θ, we have

$$\theta = p_i \mathrm{d}q^i \Longrightarrow \nabla = p_i \partial_{p_i}\,.$$

A supercrash course on supergeometry

- The superspace $\mathbb{R}^{p|q} = \mathbb{R}^p \times$ (Grassmann algebra) has canonical coordinates $(x^1, \ldots, x^p, \xi^q, \ldots, \xi^q)$, where x are commuting and ξ anti-commuting:

$$x^i \cdot x^j = x^j \cdot x^i\,, \quad x^i \cdot \xi^a = \xi^a \cdot x^i\,, \quad \xi^a \cdot \xi^b = -\xi^b \cdot \xi^a\,.$$

- Superroughly speaking, a $(p|q)$-dimensional supermanifold M is a topological space that is locally isomorphic to $\mathbb{R}^{p|q}$. There is an associated p-dimensional manifold $|M|$ called the body of M.
- In this talk, I will be interested only in local coordinates, so we can just think $M = \mathbb{R}^{p|q}$ and $|M| = \mathbb{R}^p$.
- Smooth functions on M are polynomials on the anticommuting variables ξ with functions on the commuting variables x as coefficients, e.g. in $\mathbb{R}^{p|2}$ these are of the form

$$f(x^1, \ldots x^p, \xi^1, \xi^2) = f_0(x) + f_1(x)\,\xi^1 + f_2(x)\,\xi^2 + f_{12}(x)\,\xi^1 \cdot \xi^2\,.$$

A supercrash course on supergeometry

- The fact that there are commuting and anti-commuting coordinates makes supermanifolds equipped with a $\mathbb{Z}_2$-grading.
- We call objects with $\mathbb{Z}_2$-degree 0 (resp. 1) **even** (resp. **odd**).
- A tangent vector v at a point p is defined as a superderivation on the space of functions on M:

$$v(f\cdot g)=v(f)\cdot g(p)+(-1)^{|v|\,|f|}f(p)\cdot v(g)\,,$$

where $|\cdot|$ denotes the $\mathbb{Z}_2$-grading.
- Coordinates (x^i,ξ^a) induce a basis $(\partial_{x^i},\partial_{\xi^a})$ of the tangent space T_pM at p such that

$$\partial_{x^i}(x^j)=\delta_i^j\,,\quad \partial_{\xi^a}(\xi^b)=\delta_a^b\,,\quad \partial_{x^i}(\xi^a)=0=\partial_{\xi^a}(x^i)\,.$$

- With this, it is possible to extend the notions of vector field, differential form, (co)tangent bundle, and (co)distribution.

A supercrash course on supergeometry

- The wedge product now depends on the $\mathbb{Z}_2$-grading, as well as the usual $\mathbb{N}$-grading:

$$\alpha \wedge \beta = (-1)^{kl + |\alpha|\,|\beta|} \beta \wedge \alpha\,,$$

for any k-form α and any l-form β.

Definition

A distribution $D \subseteq \mathrm{T}M$ (resp. codistribution $D \subseteq \mathrm{T}^*M$) on a homogeneity (super)manifold (M, ∇) is called a **homogeneous distribution** if the tangent lift $\mathrm{d}_{\mathrm{T}}\nabla$ (resp. the cotangent lift $\mathrm{d}_{\mathrm{T}}^*\nabla$) is tangent to D.

Theorem (Grabowska and Grabowski, 2024)

$D \subseteq \mathrm{T}M$ is a homogeneous distribution iff it is locally generated by homogeneous vector fields.

Corollary

$D \subseteq \mathrm{T}^*M$ is a homogeneous codistribution iff it is locally generated by homogeneous one-forms.

Homogeneous Frobenius theorem

Theorem (Grabowski and Grabowska, 2025)

Let $D \subset \mathrm{T}M$ be a rank-k involutive homogeneous distribution on a homogeneity supermanifold (M, ∇) of total dimension n, and let $m \in |M|$. Then, there is a neighbourhood of m endowed with a system (x^i), $i = 1, \ldots, n$, of homogeneous local coordinates such that

1. *D is locally spanned by $\langle \partial_{x^1}, \ldots, \partial_{x^k} \rangle$, if either $\nabla(m) = 0$ or $\nabla(m) \neq 0$ and $\nabla(m) \notin D_m$;*
2. *D is locally spanned by $\langle \partial_{x^1}, \ldots, \partial_{x^{k-1}}, Y \rangle$, where*

$$Y = \nabla + \sum_{j=k}^{n-1} h_j(x^k, \ldots, x^{n-1}) \partial_{x^j},$$

in the case $\nabla(m) \neq 0$ and $\nabla(m) \in D_m$.

Homogeneous symplectic Darboux theorem

Theorem (Grabowska and Grabowski, 2024)

Let ω be (λ, σ)-homogeneous symplectic form on a homogeneity (super)manifold (M, ∇) (with $\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{Z}_2$). Around every $x_0 \in |M|$ such that $\nabla(x_0) = 0$, there is a system of homogeneous coordinates (q^i, p_i, ξ^I) such that

$$\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i + \sum_I \varepsilon^I \mathrm{d}\xi^I \wedge \mathrm{d}\xi^I , \quad \varepsilon^I = \pm 1 .$$

Definition

A **presymplectic form** ω on a (super)manifold M is a closed 2-form of constant rank r. Its **characteristic distribution** $C_\omega \subseteq \mathrm{T}M$ is given by

$$C_\omega = \ker \omega \,.$$

Proposition

The characteristic distribution C_ω is an integrable distribution. Moreover, if ω is homogeneous (w.r.t. a weight vector field ∇ on M), then C_ω is a homogeneous distribution.

Homogeneous presymplectic Darboux theorem

Theorem (Grabowski and L. G.)

Let ω be (λ, σ)-homogeneous presymplectic form on a homogeneity (super)manifold (M, ∇) (with $\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{Z}_2$). Around any point $x_0 \in |M|$ such that either $\nabla(m) = 0$ or $\nabla(m) \neq 0$ and $\nabla(m) \notin \ker \omega_m$, there is a system of homogeneous coordinates $(q^i, p_i, \xi^I, z^a, \chi^b)$ such that

$$\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i + \sum_I \varepsilon^I \mathrm{d}\xi^I \wedge \mathrm{d}\xi^I, \quad \varepsilon^I = \pm 1 .$$

Class of a one-form

Definition

Let α be a k-form on a supermanifold M. The subset

$$\chi(\alpha) = \ker(\alpha) \cap \ker(\mathrm{d}\alpha) \subseteq \mathrm{T}M$$

is called the **characteristic set** of α.
If $\chi(\alpha)$ is a distribution, it is called the **characteristic distribution** of α, we say that α is **regular**, and the corank of $\chi(\alpha)$ as a sub-bundle of $\mathrm{T}M$ is called the **class of** α:

$$c_\alpha := \operatorname{corank}\left(\chi(\alpha)\right).$$

Class of a one-form

Remark

For a one-form α on a (standard) manifold M, this is equivalent to the classical definition of class, namely:

- $\operatorname{class}(\alpha) = 2s + 1$ iff $\begin{cases} \alpha \wedge \mathrm{d}\alpha^s \neq 0\,, \\ \mathrm{d}\alpha^{s+1} = 0\,. \end{cases}$
- $\operatorname{class}(\alpha) = 2s$ iff $\begin{cases} \alpha \wedge \mathrm{d}\alpha^{s-1} \neq 0\,, \\ \mathrm{d}\alpha^s \neq 0\,, \\ \alpha \wedge \mathrm{d}\alpha^s = 0\,. \end{cases}$

Class of a one-form

Proposition

If α is a regular form, then $\chi(\alpha)$ is involutive and α is $\chi(\alpha)$-invariant.

Non-degenerate one-forms

Definition

A regular one-form α on a (super)manifold M is called **non-degenerate** if its characteristic foliation is trivial:

$$\chi(\alpha) = \{0_M\}\,,$$

or equivalently,

$$c_\alpha = \dim(M)\,.$$

Non-degenerate one-forms

The annihilator of $\chi(\alpha)$ is given by

$$\big(\chi(\alpha)\big)^\circ = (\ker\alpha \cap \ker \mathrm{d}\alpha)^\circ = (\ker\alpha)^\circ + (\ker \mathrm{d}\alpha)^\circ = \langle\alpha\rangle + \mathrm{Im}(\flat_{\mathrm{d}\alpha})\,,$$

where $\flat_{\mathrm{d}\alpha}\colon \mathrm{T}M \ni v \mapsto \iota_v \mathrm{d}\alpha \in \mathrm{T}^*M$.

The form is non-degenerate iff $\big(\chi(\alpha)\big)^\circ = \mathrm{T}^*M$, so there are two possible cases for $\dim M \geq 2$:

1. $\mathrm{T}^*M = \langle\alpha\rangle \oplus \mathrm{Im}(\flat_{\mathrm{d}\alpha}) \implies c_\alpha = \mathrm{rank}(\flat_{\mathrm{d}\alpha}) + 1$ (**contact form**),
2. $\mathrm{T}^*M = \mathrm{Im}(\flat_{\mathrm{d}\alpha}) \implies c_\alpha = \mathrm{rank}(\flat_{\mathrm{d}\alpha})$ (**symplectic potential**).

Non-degenerate one-forms

The situation $\dim M = 1$, on the other hand, is trivial, since then every one-form is closed.

Remark

On a (standard) manifold, the rank of $\mathrm{d}\alpha$ is always even. Thus, a non-degenerate form α is

- a presymplectic potential iff $\dim M$ is even,
- contact iff $\dim M$ is odd.

Definition

Let α be a regular one-form on a (super)manifold M. We call α a **precontact form** (resp. a **presymplectic** potential) if the induced one-form α_{red} on $M/\chi(\alpha)$ is a **contact form** (resp. a **symplectic** potential).

Remark

If α is regular, then $\mathrm{d}\alpha$ is presymplectic.

Darboux theorem for homogeneous one-forms

Theorem (Darboux theorem for homogeneous one-forms)

Let α be a regular homogeneous one-form of degree $\lambda = (\sigma, w) \in \mathbb{Z}_2 \times \mathbb{R}$ on a homogeneity supermanifold (M, ∇). Around each point $x_0 \in |M|$ such that either $\nabla(m) = 0$ or $\nabla(m) \neq 0$ and $\nabla(m) \notin \chi(\alpha)_m$:

1. *If α is a precontact form of class $2r + s + 1$ and $\nabla(x_0) = 0$ or $w \neq 0$,*

$$\alpha = \mathrm{d}z + \sum_{i=1}^{r} p_i \mathrm{d}q^i + \sum_{l=1}^{s} \varepsilon^l y^l \mathrm{d}y^l, \quad \varepsilon^l = \pm 1\,,$$

   in a certain system of homogeneous coordinates (q^i, p_i, z, y^l, x^a) centered at x_0.

The coordinates (y^l) only appear if α is even, i.e. $\sigma = 0$.

Darboux theorem for homogeneous one-forms

Theorem (Darboux theorem for homogeneous one-forms)

Let α be a regular homogeneous one-form of degree $\lambda = (\sigma, w) \in \mathbb{Z}_2 \times \mathbb{R}$ on a homogeneity supermanifold (M, ∇). Around each point $x_0 \in |M|$ such that either $\nabla(m) = 0$ or $\nabla(m) \neq 0$ and $\nabla(m) \notin \chi(\alpha)_m$:

1. *If α is a precontact form of class $2r + s + 1$, $\nabla(x_0) \neq 0$ and $w = 0$,*

$$\alpha = \frac{\mathrm{d}z}{z} + \sum_{i=1}^{r} p_i \mathrm{d}q^i + \sum_{l=1}^{s} \varepsilon^l y^l \mathrm{d}y^l, \quad \varepsilon^l = \pm 1\,,$$

in a certain system of homogeneous coordinates (q^i, p_i, z, y^l, x^a) such that

$$z(x_0) = 1\,, \quad q^i(x_0) = p_i(x_0) = y^l(x_0) = x^a(x_0) = 0\,.$$

The coordinates (y^l) only appear if α is even, i.e. $\sigma = 0$.

Darboux theorem for homogeneous one-forms

Theorem (Darboux theorem for homogeneous one-forms)

Let α be a regular homogeneous one-form of degree $\lambda = (\sigma, w) \in \mathbb{Z}_2 \times \mathbb{R}$ on a homogeneity supermanifold (M, ∇). Around each point $x_0 \in |M|$ such that either $\nabla(m) = 0$ or $\nabla(m) \neq 0$ and $\nabla(m) \notin \chi(\alpha)_m$:

1. *If α is a presymplectic potential of class $2r + s$,*

$$\alpha = \sum_{i=1}^{r} p_i \mathrm{d}q^i + \sum_{l=1}^{s} \varepsilon^l y^l \mathrm{d}y^l, \quad \varepsilon^l = \pm 1\,,$$

in a certain system of homogeneous coordinates (q^i, p_i, y^l, x^a)

The coordinates (y^l) only appear if α is even, i.e. $\sigma = 0$.

Corollary

Let α be a regular λ-homogeneous one-form on a homogeneity manifold (M, ∇). Around each point $x_0 \in M$ such that either $\nabla(m) = 0$ or $\nabla(m) \neq 0$ and $\nabla(m) \notin \chi(\alpha)_m$, there exists a system of homogeneous coordinates in which α has a canonical expression:

1. $\alpha = \mathrm{d}z + \sum_{i=1}^{r} p_i \mathrm{d}q^i$, *if α is precontact, and $\nabla(x_0) = 0$ or $w \neq 0$,*
2. $\alpha = \frac{\mathrm{d}z}{z} + \sum_{i=1}^{r} p_i \mathrm{d}q^i$, $\nabla = z\partial_z$ *and $z(m) = 1$ if α is precontact, $\nabla(x_0) \neq 0$ and $w = 0$,*
3. $\alpha = \sum_{i} p_i \mathrm{d}x^i$, *if α is a presymplectic potential.*

Homogeneous Poincaré lemma

Lemma (Grabowska and Grabowski, 2024)

Let ω be a λ-homogeneous k-form (with $k > 0$ and $\lambda \in \mathbb{R}$) on a homogeneity (super)manifold (M, ∇). In a neighbourhood of each $x_0 \in M$ such that $\nabla(x_0) = 0$, there exists a $(k-1)$-form α such that:

1. $\mathrm{d}\alpha = \omega$,
2. *α is λ-homogeneous,*
3. $\alpha(x_0) = 0$.

Example (Non-trivial homogeneous forms of weight zero)

- Consider $\mathbb{R}^3$ with canonical coordinates (x, y, z) and $\nabla = x\partial_x - y\partial_y$.
- Any smooth functions $f_i \colon \mathbb{R}^3 \to \mathbb{R}$ of the form

$$f_i(x, y, z) = \varphi_i(xy, z)$$

  are homogeneous of weight 0.
- Let $q = x(1 + f_1)\,, \quad p = y(1 + f_2)\,, \quad \zeta = f_3$ such that

$$\left.\frac{\partial f_3}{\partial z}\right|_{(0,0,0)} \neq 0$$

- Then, $\eta = \mathrm{d}\zeta + p\mathrm{d}q$ is a (local) homogeneous contact form of weight 0.

Example (Non-trivial homogeneous forms of weight zero)

- Consider $\mathbb{R}^3$ with canonical coordinates (x, y, z) and $\nabla = x\partial_x - y\partial_y$.
- For instance,

$$\eta = y\Big(1 + \sin z + \cos(xy)(1 + \sin z) - \sin(xy)\big(e^z + xy(1 + \sin z)\big)\Big)\,\mathrm{d}x \\ - x\sin(xy)\Big(xy(\sin z + 1) + e^z\Big)\,\mathrm{d}y + e^z\cos(xy)\,\mathrm{d}z$$

  is a homogeneous contact form of weight 0.
- A system of homogeneous Darboux coordinates is

$$q = x\big(1 + \cos(xy)\big)\,, \quad p = y\big(1 + \sin(xy)\big)\,, \quad \zeta = e^z\cos(xy)\,,$$

  so that $\eta = \mathrm{d}\zeta + p\mathrm{d}q$ and $\nabla = q\partial_q - p\partial_p$.

Example (Non-trivial homogeneous forms of weight zero)

- Consider $\mathbb{R}^3$ with canonical coordinates (x, y, z) and $\nabla = x\partial_x - y\partial_y$.
- Similarly, the one-form

$$\theta = y\big(\cosh(xy) + 1\big)\big(\sinh(xy) + xy\cosh(xy) + 1\big)\,\mathrm{d}x + x^2 y\cosh(xy)\big(\cosh(xy) + 1\big)\,\mathrm{d}y$$

is homogeneous presymplectic potential of weight 0.
- Homogeneous Darboux coordinates (q, p, ζ) are given by

$$q = x\big(1 + \sinh(xy)\big)\,, \quad p = y\big(1 + \cosh(xy)\big)\,, \quad \zeta = z\,.$$

Main references

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Congratulations, Juan Carlos!

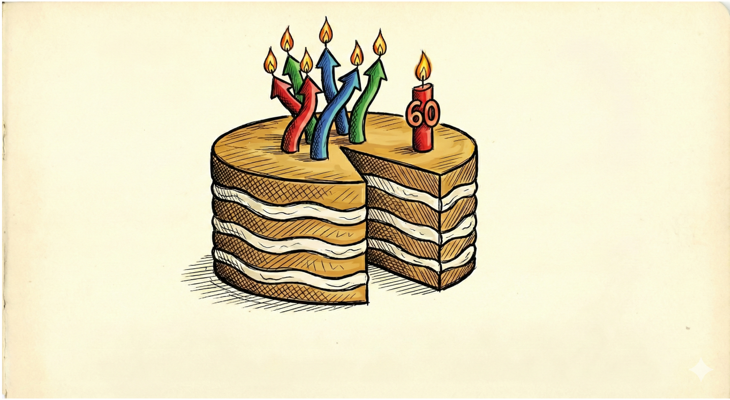

Figure: Candle bundle over a cake foliated by cream and dough leaves. Choosing your favourite flavour is left as an exercise for the audience.

Thank you for your attention!

✉ Feel free to contact me at alopez-gordon@impan.pl
🌐 These slides are available at www.alopezgordon.xyz