Liouville – Arnol'd theorem for contact Hamiltonian systems

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Symplectic geometry

- Symplectic manifolds are the natural geometric frameworks for Hamiltonian mechanics.
- Let me recall that a symplectic manifold (M, ω) is a $2n$-dimensional manifold endowed with a 2-form ω such that $\mathrm{d}\omega = 0$ and $\omega^n \neq 0$.
- The Hamiltonian vector field X_h of a function $h \in \mathscr{C}^\infty(M)$ is given by $\omega(X_h, \cdot) = 0$.
- In a neighborhood of each point in M there are canonical (or Darboux) coordinates (q^i, p_i) in which

$$\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i \,, \quad X_h = \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i}\frac{\partial}{\partial p_i} \,.$$

Liouville–Arnol'd theorem

Theorem (Liouville–Arnol'd)

Let $f_1, \ldots, f_n$ be independent functions in involution (i.e., $\{f_i, f_j\} = 0\ \forall i, j$) on a symplectic manifold (M^{2n}, ω). Let $M_\Lambda = \{x \in M \mid f_i = \Lambda_i\}$ be a regular level set.

1. *Any compact connected component of M_Λ is diffeomorphic to $\mathbb{T}^n$.*
2. *On a neighborhood of M_Λ there are coordinates (φ^i, J_i) such that*

$$\omega = \mathrm{d}\varphi^i \wedge \mathrm{d}J_i\,,$$

and $f_i = f_i(J_1, \ldots, J_n)$, so the Hamiltonian vector fields read

$$X_{f_i} = \frac{\partial f_i}{\partial J_j} \frac{\partial}{\partial \varphi^j}\,.$$

Liouville–Arnol'd theorem

Corollary

Let (M^{2n}, ω, h) *be a Hamiltonian system. Suppose that* $f_1, \ldots, f_n$ *are independent conserved quantities (i.e.* $X_h(f_i) = 0\ \forall\, i$*) in involution. Then, on a neighborhood of* M_Λ *there are Darboux coordinates* (φ^i, J_i) *such that* $h = h(J_1, \ldots, J_n)$*, so the Hamiltonian dynamics are given by*

$$\frac{\mathrm{d}\varphi^i}{\mathrm{d}t} = \frac{\partial h}{\partial J_i}\frac{\partial}{\partial \varphi^i}\,,$$

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = 0\,.$$

Example (The n-dimensional harmonic oscillator)

- Consider $\mathbb{R}^{2n}$, with canonical coordinates (x_i, p_i), $i \in \{1, \ldots, n\}$, equipped with the symplectic form ω and the Hamiltonian function h,

$$\omega = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}p_i \,, \quad h = \sum_{i=1}^{n} \left(\frac{p_i^2}{2} + \frac{x_i^2}{2} \right)$$

- The functions $f_i = \frac{p_i^2}{2} + \frac{x_i^2}{2}$ are independent and involution, and one can write $h = \sum_{i=1}^{n} f_i$.
- Angle coordinates are $\varphi^i = \arctan\left(\frac{x_i}{p_i}\right)$ and action coordinates are f_i.
- Hamilton's equations read

$$\frac{\mathrm{d}\varphi^i}{\mathrm{d}t} = 1 \,, \qquad \frac{\mathrm{d}f_i}{\mathrm{d}t} = 0 \,.$$

Integrable distributions

- Given a differentiable manifold M, a **distribution** D of (co)rank k on M is a subbundle of the tangent bundle $\mathrm{T}M$, i.e., a smoooth assignment of a k-(co)dimensional vector subspace $D_x \subseteq \mathrm{T}_x M$ for each $x \in M$.

Theorem (Frobenius)

The following statements are equivalent:

1. *For every $x \in M$, there exists a submanifold $N \subseteq M$ such that $D_x = \mathrm{T}_x N$ (i.e., D is **integrable**).*
2. *For each pair of vector fields $X, Y \in \mathfrak{X}(M)$ such that $X(x), Y(x) \in D_x$ for all $x \in M$ we have that $[X, Y](x) \in D_x$ (i.e., D is **involutive**).*

Maximally non-integrable distributions

- *Grosso modo*, a distribution D will be "as far as possible" from being integrable if

$$X, Y \in D \Longrightarrow [X, Y] \notin D \text{ or } [X, Y] = 0\,.$$

- More precisely, we will say that D is **maximally non-integrable** if the bilinear map

$$\nu_D \colon D \times_M D \ni (X, Y) \mapsto \gamma\Big([X, Y]\Big) \in \mathrm{T}M/D$$

is non-degenerate. Here $[\cdot, \cdot]$ denotes the Lie bracket of vector fields with image in D, and $\gamma \colon \mathrm{T}M \to \mathrm{T}M/D$ is the canonical projection.

Contact distributions

Definition

Let M be a $(2n+1)$-dimensional manifold. A **contact distribution** C on M is a maximally non-integrable distribution of corank 1. The pair (M, C) is called a **contact manifold**.

Distributions as kernels of 1-forms

- Note that a codistribution D of corank 1 on M can be locally written as the kernel of a (local) 1-form α on M.
- It is easy to see that D is integrable iff

$$\alpha \wedge \mathrm{d}\alpha = 0$$

  for any local 1-form α such that $D = \ker \alpha$.
- On the contrary, D is maximally non-integrable iff

$$\alpha \wedge \mathrm{d}\alpha^n = \alpha \wedge \underbrace{\mathrm{d}\alpha \wedge \cdots \wedge \mathrm{d}\alpha}_{n \text{ times}} \neq 0$$

  for any local 1-form α such that $D = \ker \alpha$.

Contact forms

Definition

Let (M, C) be a contact manifold such that C can be globally written as the kernel of a global 1-form η on M. Then, C is said to be a **co-orientable** contact distribution, η is called a **contact form**, and the pair (M, η) is called a **co-oriented contact manifold**.

Contact forms

Remarks

- A co-orientable contact distribution C does not fix the contact form η, but rather the equivalence class

$$\eta \sim \tilde{\eta} \Longleftrightarrow \ker \eta = \ker \tilde{\eta} \Longleftrightarrow \exists\, f \colon M \to \mathbb{R} \setminus \{0\} \text{ such that } \tilde{\eta} = f\eta\,.$$

- Not all contact manifolds are co-orientable. Nevertheless, their double cover is always co-orientable.
- Several authors refer to co-oriented contact manifolds as contact manifolds. The term "contact structure" is used to refer either to the contact distribution or to the contact form, so I will not use it in order to avoid ambiguity.

Example (Odd-dimensional Euclidean space)

$\eta = \mathrm{d}z - \sum y^i \mathrm{d}x^i$, in $\mathbb{R}^{2n+1}$ with canonical coordinates (x^i, y^i, z).

Example (Trivial bundle over the cotangent bundle)

The cotangent bundle T^*Q of Q is endowed with the tautological 1-form θ_Q. The trivial bundle $\pi_1\colon \mathrm{T}^*Q \times \mathbb{R} \to \mathrm{T}^*Q$ can be equipped with the contact form $\eta_Q = \mathrm{d}r - \pi^*\theta_Q$, with r the canonical coordinate of $\mathbb{R}$. If (q^i) are coordinates in Q which induce bundle coordinates (q^i, p_i) in T^*Q and (q^i, p_i, r) in $\mathrm{T}^*Q \times \mathbb{R}$, we have

$$\theta_Q = p_i \mathrm{d}q^i\,, \quad \eta_Q = \mathrm{d}r - p_i \mathrm{d}q^i\,.$$

Example (Projective space)

Let $M = \mathbb{R}^n \times \mathbb{RP}^{n-1}$. Consider the open subsets

$$U_k = \{(x, [y]) \in M \mid y^k \neq 0\} ,$$

where $x = (x^1, \ldots, x^n), y = (y^1, \ldots, y^k, \ldots, y^n) \in \mathbb{R}^n$. We have the local contact forms

$$\eta_k = \mathrm{d}x^k - \sum_{i \neq k} \frac{y_i}{y_k} \mathrm{d}x^i \in \Omega^1(U_k) .$$

If a global contact form η on M existed, then $\eta \wedge \mathrm{d}\eta^n$ would define an orientation. Hence, M is not co-orientable if n is even.

The Reeb vector field

Definition

Let (M, η) be a co-oriented contact manifold. The **Reeb vector field** of (M, η) is the unique vector field $\mathcal{R} \in X(M)$ such that

$$\mathcal{R} \in \ker \mathrm{d}\eta\,, \qquad \eta(\mathcal{R}) = 1\,.$$

The tangent bundle $\mathsf{T}M$ of a co-oriented contact manifold (M, η) can be decomposed as the Whitney sum

$$\mathsf{T}M = \ker \eta \oplus \ker \mathrm{d}\eta = C \oplus \langle \mathcal{R} \rangle \,.$$

Note that the complement of the contact distribution $C = \ker \eta$ depends on the choice of contact form.

Proposition

Let η be a 1-form on a manifold M. The map

$$\flat_\eta \colon \mathfrak{X}(M) \to \Omega^1(M)\,, \quad \flat_\eta(X) = \eta(X)\eta + \iota_X \mathrm{d}\eta$$

is a $\mathscr{C}^\infty(M)$-module isomorphism iff η is a contact form.

Note that the Reeb vector field can be equivalently defined as $\mathcal{R} = \flat_\eta^{-1}(\eta)$.

Darboux coordinates

Theorem

Let (M, η) be a $(2n+1)$-dimensional co-oriented contact manifold. Around each point $x \in M$ there exist local coordinates (q^i, p_i, z), $i \in \{1\ldots, n\}$ such that the contact form reads

$$\eta = \mathrm{d}z - p_i \mathrm{d}q^i \,.$$

Consequently, the Reeb vector field is written as

$$\mathcal{R} = \frac{\partial}{\partial z} \,.$$

These coordinates are called ***canonical*** *or* ***Darboux*** *coordinates.*

Jacobi structures

- Consider a manifold M endowed with a bivector field $\Lambda \in \mathrm{Sec}(\bigwedge^2 \mathrm{T}M)$ and a vector field $E \in \mathfrak{X}(M)$.
- Define the bracket $\{\cdot,\cdot\}\colon \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ by

$$\{f,g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + fE(g) - gE(f)\,.$$

- It is a Lie bracket iff

$$[\Lambda, E] = 0\,, \quad [\Lambda, \Lambda] = 2E \wedge \Lambda\,,$$

  where $[\cdot,\cdot]$ denotes the Schouten–Nijenhuis bracket.
- In that case, (Λ, E) is called a **Jacobi structure** on M, $\{\cdot,\cdot\}$ is called a Jacobi bracket, and (M, Λ, E) is called a Jacobi manifold.

Jacobi structures

Remark

A Poisson structure Λ is a Jacobi structure with $E \equiv 0$.

Jacobi structures

- A Jacobi structure (Λ, E) defines a $\mathscr{C}^\infty(M)$-module morphism

$$\sharp_\Lambda \colon \Omega^1(M) \to \mathfrak{X}(M)\,, \qquad \sharp_\Lambda(\alpha) = \Lambda(\alpha, \cdot)\,.$$

- This defines a so-called orthogonal complement $D^{\perp_\Lambda} = \sharp_\Lambda(D^\circ)$, for a distribution D with annihilator D°.
- A submanifold N of M is called **coisotropic** if $\mathrm{T}N^{\perp_\Lambda} \subseteq \mathrm{T}N$.

Jacobi structures

- Two Jacobi structures (Λ, E) and $(\tilde{\Lambda}, \tilde{E})$ on M are **conformally equivalent** if there exists a nowhere-vanishing function f on M such that
$$\tilde{\Lambda} = f\Lambda\,, \quad \tilde{E} = \sharp_\Lambda \mathrm{d}f + fE\,.$$

Remark

The orthogonal complement coincides for conformally equivalent Jacobi structures, namely, $D^{\perp_\Lambda} = D^{\perp_{\tilde{\Lambda}}}$ for any distribution D.

Jacobi structures

Definition

Let (M, Λ, E) be a Jacobi manifold with Jacobi bracket $\{\cdot, \cdot\}$. A collection of functions $f_1, \ldots, f_k \in \mathscr{C}^\infty(M)$ will be said to be **in involution** if

$$\{f_i, f_j\} = 0 \, , \forall \, i, j \in \{1, \ldots, k\} \, .$$

Jacobi structures

- For each function $f \in \mathscr{C}^\infty(M)$, we can define a vector field

$$X_f = \sharp_\Lambda(\mathrm{d}f) + fE\,,$$

or, equivalently,

$$X_f(g) = \{f, g\} + gE(f)\,, \quad \forall g \in \mathscr{C}^\infty(M)\,.$$

- Following the nomenclature of Dazord, Lichnerowicz, Marle, *et al.*, we will refer to X_f as the **Hamiltonian vector field of** f.
- However, X_f does not satisfy the properties of a usual Hamiltonian vector field (w.r.t. a symplectic or Poisson structure). In particular,

$$\{f, g\} = 0 \not\Longleftrightarrow X_f(g) = 0\,.$$

Jacobi structure defined by a contact form

- A co-oriented contact manifold (M^{2n+1}, η) is endowed with a Jacobi structure (Λ, E) given by

$$\Lambda(\alpha, \beta) = -\mathrm{d}\eta\Big(\flat_\eta^{-1}(\alpha), \flat_\eta^{-1}(\beta)\Big)\,, \quad E = -\mathcal{R}\,,$$

  where $\mathcal{R}$ is the Reeb vector field.
- Any contact form $\tilde{\eta}$ defining the same contact distribution, i.e., $\ker \tilde{\eta} = \ker \eta$, defines a conformally equivalent Jacobi structure.

Contact Hamiltonian vector field

- Let (M, η) be a co-oriented contact manifold. The Hamiltonian vector field of $f \in \mathscr{C}^\infty(M)$ is uniquely determined by

$$\eta(X_f) = -f\,, \quad \mathcal{L}_{X_f}\eta = -\mathcal{R}(f)\eta\,.$$

- In Darboux coordinates

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i\frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z}\,.$$

Contact Hamiltonian vector field

Remarks

- The Reeb vector field is the Hamiltonian vector field of $f \equiv -1$.
- Every Hamiltonian vector field is an infinitesimal contactomorphism (i.e., its flow preserves the contact distribution $C = \ker \eta$). Conversely, if $Y \in \mathfrak{X}(M)$ is an infinitesimal contactomorphism, then it is the Hamiltonian vector field of $f = -\eta(Y)$.
- Knowing $C = \ker \eta$ and X_f does not fix η nor f. As a matter of fact, X_f is the Hamiltonian vector field of $g = f/a$ with respect to $\tilde{\eta} = a\eta$, for any non-vanishing $a \in \mathscr{C}^\infty(M)$.

Contact Hamiltonian systems

Definition

A **contact Hamiltonian system** (M, η, h) is a co-oriented contact manifold (M, η) with a fixed **Hamiltonian function** $h \in \mathscr{C}^\infty(M)$.

- The dynamics of (M, η, h) is determined by the integral curves of the Hamiltonian vector field X_h of h w.r.t. η.

Contact Hamiltonian systems

- In Darboux coordinates, these curves $c(t) = (q^i(t), p_i(t), z(t))$ are determined by the **contact Hamilton equations**:

$$\frac{\mathrm{d}q^i(t)}{\mathrm{d}t} = \frac{\partial h}{\partial p_i} \circ c(t)\,,$$

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = -\frac{\partial h}{\partial q^i} \circ c(t) - p_i(t)\frac{\partial h}{\partial z} \circ c(t)\,,$$

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = p_i(t)\frac{\partial h}{\partial p_i} \circ c(t) - h \circ c(t)\,.$$

Example (The harmonic oscillator with linear damping)

Consider the solution $x\colon \mathbb{R} \to \mathbb{R}$ of the second-order ordinary differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t) = -x(t) - \kappa \frac{\mathrm{d}x}{\mathrm{d}t}(t)\,,$$

where $\kappa \in \mathbb{R}$. Defining $p = \mathrm{d}x/\mathrm{d}t$, we can reduce it to the system of first-order ordinary differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = p(t)\,, \quad \frac{\mathrm{d}p}{\mathrm{d}t}(t) = -x(t) - \kappa p(t)\,.$$

We can obtain this system as the two first contact Hamilton equations from the contact Hamilton system $(\mathbb{R}^3, \eta, h)$, where $\eta = \mathrm{d}z - p\mathrm{d}x$ and

$$h = \frac{p^2}{2} + \frac{x^2}{2} + \kappa z\,.$$

Notions of integrability for contact manifolds

- Khesin and Tabachnikov, Liberman, Banyaga and Molino, Lerman, etc. have defined notions of contact complete integrability which are geometric but not dynamical, e.g. a certain foliation over a contact manifold.
- Boyer considers the so-called good Hamiltonians h, i.e., $\mathcal{R}(h) = 0 \rightsquigarrow$ no dissipation, "symplectic" dynamics.
- Miranda considered integrability of the Reeb dynamics when $\mathcal{R}$ is the generator of an $\mathbb{S}^1$-action.
- We are interested in complete integrability of contact Hamiltonian dynamics.

Notions of integrability for contact manifolds

Instead, we will use the equivalence between the categories of contact manifolds and symplectic $\mathbb{R}^\times$-principal bundles, and proof a Liouville–Arnol'd theorem for homogeneous functions in involution.

Exact symplectic manifolds

Definition

An **exact symplectic manifold** is a pair (M, θ), where θ is a **symplectic potential** on M, i.e., $\omega = -\mathrm{d}\theta$ is a symplectic form on M. The **Liouville vector field** $\nabla \in \mathfrak{X}(M)$ is given by

$$\iota_\nabla \omega = -\theta\,.$$

A tensor field A on P is called k-homogeneous (for $k \in \mathbb{Z}$) if

$$\mathcal{L}_\nabla A = kA\,.$$

Homogeneous integrable system

Definition

A **homogeneous integrable system** consists of an exact symplectic manifold (M^{2n}, θ) and a map $F = (f_1, \dots, f_n) \colon M \to \mathbb{R}^n$ such that the functions $f_1, \dots, f_n$ are independent, in involution and homogeneous of degree 1 (w.r.t. the Liouville vector field ∇ of θ) on a dense open subset $M_0 \subseteq M$. We will denote it by (M, θ, F).

For simplicity's sake, in this talk I will assume that $M_0 = M$.

Proposition

Let (M, θ, F) *be a homogeneous integrable system. Then, for each* $\Lambda \in \mathbb{R}^n$*, the level set* $M_\Lambda = F^{-1}(\Lambda)$ *is a Lagrangian submanifold, and*

$$\phi_t^\nabla(M_\Lambda) = M_{t\Lambda} = F^{-1}(t\Lambda)\,,$$

where ϕ_t^∇ *denotes the flow of the Liouville vector field* ∇*.*

- Consider the exact symplectic manifold (M, θ), with Liouville vector field ∇.
- Around each point in M, there are canonical coordinates (q^i, p_i) where $\theta = p_i \mathrm{d}q^i$.
- Then, a straightforward computation shows that $\nabla = p_i \frac{\partial}{\partial p_i}$.
- Note that coordinates may be canonical for $\omega = -\mathrm{d}\theta$ but not for θ. For instance, in the coordinates $\tilde{q}^i = q^i$, $\tilde{p}_i = p_i + e^{q_i}$ we have

$$\theta = \sum_i (\tilde{p}_i - e^{\tilde{q}^i})\mathrm{d}\tilde{q}^i \,, \quad \omega = \mathrm{d}\tilde{q}^i \wedge \mathrm{d}\tilde{p}_i \,, \quad \nabla = \left(\tilde{p}_i - e^{\tilde{q}^i}\right) \frac{\partial}{\partial \tilde{p}_i} \,.$$

- In particular, the Liouville–Arnol'd theorem provides coordinates which are canonical for ω, but not necessarily for θ or ∇.

Homogeneous Liouville – Arnol'd theorem

Consider a homogeneous integrable system (M, θ, F). Let U be an open neighbourhood of the level set $M_\Lambda = F^{-1}(\Lambda)$ (with $\Lambda \in \mathbb{R}^n$) such that:

1. $f_1, \ldots, f_n$ have no critical points in U,
2. the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$ are complete,
3. the submersion $F \colon U \to \mathbb{R}^n$ is a trivial bundle over a domain $V \subseteq \mathbb{R}^n$.

Homogeneous Liouville – Arnol'd theorem

Theorem (Colombo, de León, Lainz, L. G., 2023)

Let (M, θ, F) be a homogeneous integrable system with $F = (f_1, \ldots, f_n)$. Given $\Lambda \in \mathbb{R}^n$, suppose that $M_\Lambda = F^{-1}(\Lambda)$ is connected, and assume the statements from the previous slide. Then, $U \cong \mathbb{T}^k \times \mathbb{R}^{n-k} \times V$ and there is a chart $(\hat{U} \subseteq U; y^i, A_i)$ of M s.t.

1. *$A_i = M_i^j f_j$, where M_i^j are homogeneous functions of degree 0 depending only on $f_1, \ldots, f_n$,*
2. *$\theta = A_i \mathrm{d} y^i$,*
3. *$X_{f_i} = N_i^j \frac{\partial}{\partial y^j}$, with (N_i^j) the inverse matrix of (M_i^j).*

Lemma

Let M be an n-dimensional manifold, and let $X_1, \dots, X_n \in \mathfrak{X}(M)$ be linearly independent vector fields. If these vector fields are pairwise commutative and complete, then M is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$ for some $k \leq n$, where $\mathbb{T}^k$ denotes the k-dimensional torus.

Lemma

Let (M^{2n}, θ, F) *be a homogeneous integrable system, with* $F = (f_1, \dots, f_n)$*. Assume that the Hamiltonian vector fields* X_{f_i} *are complete. Then, there exists n functions* $g_i = M_i^j f_j \in \mathscr{C}^\infty(M)$ *such that*

1. $\big(M, \theta, (g_1, \dots, g_n)\big)$ *is also a homogeneous integrable system,*
2. $X_{g_1}, \dots, X_{g_k}$ *are infinitesimal generators of* $\mathbb{S}^1$*-actions and their flows have period 1,*
3. $X_{g_{k+1}}, \dots, X_{g_n}$ *are infinitesimal generators of* $\mathbb{R}$*-actions,*
4. M_i^j *for* $i, j \in 1, \dots, n$ *are homogeneous functions of degree* 0*, and they depend only on* $f_1, \dots, f_n$*.*

Lemma

Let $\pi\colon P \to M$ be a G-principal bundle over a connected and simply connected manifold. Suppose there exists a connection one-form A such that the horizontal distribution H is integrable. Then $\pi\colon P \to M$ is a trivial bundle and there exists a global section $\chi\colon M \to P$ such that $\chi^* A = 0$.

Proof of the theorem

- We know that M_Λ is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$.
- W.l.o.g., assume that $X_{f_1}, \dots, X_{f_k}$ are infinitesimal generators of $\mathbb{S}^1$-actions with period 1, and that $X_{g_{k+1}}, \dots, X_{g_n}$ are infinitesimal generators of $\mathbb{R}$-actions.
- Let $\mathfrak{L} = \ker \theta$ and $\overline{U} = \{x \in U \mid f_i(x) \neq 0 \ \forall\, i \text{ and } \theta(x) \neq 0\}$.
- Since $F\colon U \to V$ is a trivial bundle, $U \cong V \times \mathbb{T}^k \times \mathbb{R}^{n-k}$ can be endowed with a Riemannian metric g, given by the product of flat metrics in $V \subseteq \mathbb{R}^n$, $\mathbb{T}^k$ and $\mathbb{R}^{n-k}$, which is flat and invariant by the Lie group action of $\mathbb{T}^k \times \mathbb{R}^{n-k}$.

Proof of the theorem

- The distribution
$$\mathfrak{L}^\theta = \left(\mathfrak{L} \cap \langle X_{f_i} \rangle_{i=1}^n\right)^{\perp_g} \cap \mathfrak{L}$$
is
  1. invariant by the Lie group action of $\mathbb{T}^k \times \mathbb{R}^{n-k}$,
  2. contained in $\mathfrak{L}$,
  3. complementary to the vertical bundle:
$$\mathfrak{L}_x^\theta \oplus \langle X_{f_i}(x) \rangle_{i=1}^n = \mathrm{T}_x M \,, \quad \forall x \in \overline{U} \,.$$
- Moreover, $F\colon \overline{U} \to \overline{U}/(\mathbb{T}^k \times \mathbb{R}^{n-k})$ is a principal bundle and $\mathfrak{L}^\theta$ is a principal connection with connection one-form θ.
- The fact that $\theta \wedge \mathrm{d}\theta = 0$ implies that $\mathfrak{L}$ is integrable.
- Since it is the orthogonal complement of $\mathfrak{L}$ w.r.t. a flat metric, $\mathfrak{L}^\theta$ is integrable.

Proof of the theorem

- Let $\hat{U} \subseteq \overline{U}$ be an open subset of $\overline{U}$ such that $\hat{V} = F(\hat{U})$ is simply connected.
- Then, there exists a global section χ of $F\colon \hat{U} \to \hat{V} \cong \hat{U}/(\mathbb{T}^k \times \mathbb{R}^{n-k})$ such that $\chi^*\theta = 0$.
- Let $\Phi\colon \mathbb{T}^k \times \mathbb{R}^{n-k} \times M \to M$ denote the action defined by the flows of X_{f_i}.
- For each point $x \in M_\Lambda = F^{-1}(\Lambda)$, the angle coordinates $(y^i(x))$ are determined by
$$\Phi(y^i(x), \chi(F(x))) = x\,.$$
- Notice that (y^i, f_i) are coordinates in $\hat{U}$ adapted to the foliation of M in M_Λ.

Proof of the theorem

- In these coordinates,

$$\theta = A_i \mathrm{d}y^i + B^i \mathrm{d}f_j \,, \quad X_{f_i} = \frac{\partial}{\partial y^i} \,,$$

- Contracting θ with X_{f_i} yields $A_i = f_i$.
- Finally, notice that $\operatorname{Im} \chi = \cap_{i=1}^n (y^i)^{-1}(\mu_i)$. Hence,

$$0 = \chi^* \theta = B^i \mathrm{d}f_i \,.$$

- Since μ_i's are arbitrary values of y^i, the functions B^i are identically zero on all the manifold M and $\theta = f_i \mathrm{d}y^i$.

□

Construction of action-angle coordinates

In order to construct action-angle coordinates in a neighbourhood U of M_Λ, one has to carry out the following steps:

1. Fix a section χ of $F\colon U \to V$ such that $\chi^*\theta = 0$.
2. Compute the flows $\phi_t^{X_{f_i}}$ of the Hamiltonian vector fields X_{f_i}.
3. Let $\Phi\colon \mathbb{R}^n \times M \to M$ denote the action of $\mathbb{R}^n$ on M defined by the flows, namely,

$$\Phi(t_1, \ldots, t_n; x) = \phi_{t_1}^{X_{f_1}} \circ \cdots \circ \phi_{t_n}^{X_{f_n}}(x) .$$

4. It is well-known that the isotropy subgroup $G_{\chi(\Lambda)(\Lambda)} = \{g \in \mathbb{R}^n \mid \Phi(g, \chi(\Lambda)) = \chi(\Lambda)\}$, forms a lattice (that is, a $\mathbb{Z}$-submodule of $\mathbb{R}^n$). Pick a $\mathbb{Z}$-basis $\{e_1, \ldots, e_m\}$, where m is the rank of the isotropy subgroup.

Construction of action-angle coordinates

5. Complete it to a basis $\mathcal{B} = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ of $\mathbb{R}^n$.
6. Let (M_i^j) denote the matrix of change from the basis $\{X_{f_i}(\chi(\Lambda))\}$ of $\mathrm{T}_{\chi(\Lambda)} M_\Lambda \simeq \mathbb{R}^n$ to the basis $\{e_i\}$. The action coordinates are the functions $A_i = M_i^j f_j$.
7. The angle coordinates (y^i) of a point $x \in M$ are the solutions of the equation

$$x = \Phi\left(y^i e_i; \chi \circ F(x)\right).$$

Trivial symplectization of a co-oriented contact manifold

Definition

Let (M, η) be a co-oriented contact manifold. Then, the trivial bundle $\pi_1 \colon M^{\mathrm{symp}} = M \times \mathbb{R}_+ \to M$, $\pi_1(x, r) = x$ can be endowed with the symplectic potential $\theta(x, r) = r\eta(x)$. The Liouville vector field reads $\nabla = r\partial_r$.

We will refer to $(M^{\mathrm{symp}}, \theta)$ as the **trivial symplectization** of (M, η).

Remark

I will present a more general setting at the end of the talk.

Trivial symplectization of a co-oriented contact manifold

Proposition

There is a one-to-one correspondence between functions $f(x)$ *on* M *and 1-homogeneous functions* $f^{\mathrm{symp}}(x, r) = -rf(x)$ *on* M^{symp} *such that the symplectic* $X_{f^{\mathrm{symp}}}$ *and contact* X_f *Hamiltonian vector fields are related as follows:*

$$\mathrm{T}\pi_1\left(X_{f^{\mathrm{symp}}}\right) = X_f\,.$$

Moreover, the Poisson $\{\cdot,\cdot\}_\theta$ *and Jacobi* $\{\cdot,\cdot\}$ *brackets have the correspondence*

$$\{f^{\mathrm{symp}}, g^{\mathrm{symp}}\}_\omega = \left(\{f, g\}_\eta\right)^{\mathrm{symp}}\,.$$

Definition

A **completely integrable contact system** is a triple (M, η, F), where (M^{2n+1}, η) is a co-oriented contact manifold and $F = (f_0, \ldots, f_n)\colon M \to \mathbb{R}^{n+1}$ is a map such that

1. $f_0, \ldots, f_n$ are in involution, i.e., $\{f_\alpha, f_\beta\} = 0 \;\forall\, \alpha, \beta \in \{0, \ldots, n\}$,
2. $\operatorname{rank} \mathrm{T}F \geq n$ on a dense open subset $M_0 \subseteq M$.

Proposition

Let (M, η) be a co-oriented contact manifold and $F\colon M \to \mathbb{R}^{n+1}$ a smooth map. Consider the trivial symplectization, i.e., $M^{\mathrm{symp}} = M \times \mathbb{R}_+$ endowed with the symplectic potential $\theta(x, r) = r\eta(x)$, and the map $F^{\mathrm{symp}}(x, r) = -rF(x)$. Then, $(M^{\mathrm{symp}}, \theta, F^{\mathrm{symp}})$ is a homogeneous integrable system iff (M, η, F) is a completely integrable contact system.

Some notation

- For each $\Lambda \in \mathbb{R}^{n+1} \setminus \{0\}$, let $\langle\Lambda\rangle_+$ denote the ray generated by Λ, namely,
$$\langle\Lambda\rangle_+ := \left\{x \in \mathbb{R}^{n+1} \mid \exists \in \mathbb{R}_+ : x = r\Lambda\right\} .$$
- Consider the preimages $M_{\langle\Lambda\rangle_+}$ of those rays by a map $F \colon M \to \mathbb{R}^{n+1}$, namely,
$$M_{\langle\Lambda\rangle_+} := F^{-1}\left(\langle\Lambda\rangle_+\right) .$$

Assumptions

1. Assume that the Hamiltonian vector fields $X_{f_0}, \dots, X_{f_n}$ are complete.
2. Given $\Lambda \in \mathbb{R}^{n+1} \setminus \{0\}$, let $B \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be an open neighbourhood of Λ.
3. Let $\pi \colon U \to M_{\langle\Lambda\rangle_+}$ be a tubular neighbourhood of $M_{\langle\Lambda\rangle_+}$ such that $F|_U \colon U \to B$ is a trivial bundle over a domain $V \subseteq B$.

Theorem (Colombo, de León, Lainz, L. G., 2023)

Let (M, η, F) be a completely integrable contact system, where $F = (f_0, \ldots, f_n)$. Consider the assumptions of the previous slide. Then:

1. $M_{\langle\Lambda\rangle_+}$ *is coisotropic, invariant by the Hamiltonian flow of* f_α*, and diffeomorphic to* $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$ *for some* $k \leq n$.
2. *There exist coordinates* $(y^0, \ldots, y^n, \tilde{A}_1, \ldots, \tilde{A}_n)$ *on* U *such that the Hamiltonian vector fields of the functions* f_α *read*

$$X_{f_\alpha} = \overline{N}_\alpha^\beta X_{f_\beta} \,,$$

where $\overline{N}_\alpha^\beta$ *are functions depending only on* $\tilde{A}_1, \ldots, \tilde{A}_n$.

3. *There exists a nowhere-vanishing function* $A_0 \in \mathscr{C}^\infty(U)$ *and a conformally equivalent contact form* $\tilde{\eta} = \eta / A_0$ *such that* $(y^i, \tilde{A}_i, y^0)$ *are Darboux coordinates for* $(M, \tilde{\eta})$*, namely,* $\tilde{\eta} = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i$.

Sketch of the proof

1. Translate the problem to the exact symplectic manifold $(M^{\mathrm{symp}} = M \times \mathbb{R}_+, \theta = r\eta)$.
   - $\{f_\alpha, f_\beta\} = 0 \Rightarrow \{f_\alpha^{\mathrm{symp}}, f_\beta^{\mathrm{symp}}\} = 0$.
   - X_{f_α} complete $\Rightarrow X_{f_\alpha^{\mathrm{symp}}}$ complete.
   - $\operatorname{rank} \mathrm{d} f_\alpha \geq n \Rightarrow \operatorname{rank} \mathrm{d}(\underbrace{r\pi_1^* f_\alpha}_{f_\alpha^{\mathrm{symp}}}) \geq n+1$.
   - $\pi_1\left((F^{\mathrm{symp}})^{-1}(\Lambda)\right) = \left\{x \in M \mid \exists s \in \mathbb{R}^+ \colon\ F(x) = \frac{\Lambda}{s}\right\} = M_{\langle\Lambda\rangle_+}$.
   - $X_{f_\alpha^{\mathrm{symp}}}$ are tangent to $(F^{\mathrm{symp}})^{-1}(\Lambda) \Rightarrow X_{f_\alpha}$ are tangent to $M_{\langle\Lambda\rangle_+}$.
   - X_{f_α} commute and are tangent to $M_{\langle\Lambda\rangle_+} \Rightarrow M_{\langle\Lambda\rangle_+} \simeq \mathbb{T}^k \times \mathbb{R}^{n+1-k}$.
   - $F \colon U \to B$ is a trivial bundle $\Rightarrow F^{\mathrm{symp}} \colon \pi_1^{-1} U \to B$ is a trivial bundle.

   $\therefore$ We can apply the theorem for exact symplectic manifolds to obtain action-angle coordinates $(y^\alpha_{\mathrm{symp}}, A_\alpha^{\mathrm{symp}})$ on $\pi_1^{-1}(U)$.

Sketch of the proof

2. In these coordinates,

$$\theta = A_\alpha^{\mathrm{symp}} \mathrm{d} y_{\mathrm{symp}}^\alpha \,, \qquad A_\alpha^{\mathrm{symp}} = M_\alpha^\beta f_\beta^{\mathrm{symp}} \,,$$

and

$$X_{f_\alpha^{\mathrm{symp}}} = N_\alpha^\beta \frac{\partial}{\partial y_{\mathrm{symp}}^\beta} \,, \quad (N_\beta^\alpha) = (M_\beta^\alpha)^{-1} \,.$$

Due to the homogeneity, there are functions $y^\alpha, A_\alpha, \overline{M}_\alpha^\beta$ and $\overline{N}_\alpha^\beta$ on M such that

$$A_\alpha^{\mathrm{symp}} = -r\left(\pi_1^* A_\alpha\right) \,, \qquad y_{\mathrm{symp}}^\alpha = \pi_1^* y^\alpha \,,$$
$$M_\alpha^\beta = \pi_1^* \overline{M}_\alpha^\beta \,, \qquad N_\alpha^\beta = \pi_1^* \overline{N}_\alpha^\beta \,.$$

Sketch of the proof

3. Since $r\left(\pi_1^* \eta\right) = \theta$, the contact form is given by

$$\eta = A_\alpha \mathrm{d} y^\alpha \,.$$

and

$$f_\alpha = \overline{M}_\alpha^\beta A_\beta \,, \quad X_{f_\alpha} = \overline{N}_\alpha^\beta \frac{\partial}{\partial y^\beta} \,,$$

4. Since $\Lambda \neq 0$, there is at least one nonvanishing f_α. Hence, there is at least one nonvanishing A_α. W.l.o.g. , assume that $A_0 \neq 0$. Then, $(y^i, \tilde{A}_i = -A_i/A_0, y^0)$ are Darboux coordinates for

$$\tilde{\eta} = \frac{1}{A_0} \eta = \mathrm{d} y^0 - \tilde{A}_i \mathrm{d} y^i \,,$$

An example

- Let $M = \mathbb{R}^3 \setminus \{0\}$ with canonical coordinates (q, p, z), and $\eta = \mathrm{d}z - p\mathrm{d}q$.
- The functions $h = p$ and $f = z$ are in involution.
- Let $F = (h, f)\colon M \to \mathbb{R}^2$.
- $\operatorname{rank} \mathsf{T}F = 2$, and thus (M, η, F) is a completely integrable contact system.

An example

- Hypothesis of the theorem are satisfied:
  1. The Hamiltonian vector fields
  $$X_h = \frac{\partial}{\partial q}\,, \quad X_f = -p\frac{\partial}{\partial p} - z\frac{\partial}{\partial z}$$
  are complete,
  2. Since $F\colon (q,p,z) \mapsto (p,z)$ is the canonical projection, $F\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ is a trivial bundle.

An example

- Therefore, $\theta = r\mathrm{d}z - rp\mathrm{d}q$ is the symplectic potential on $M^{\mathrm{symp}} = M \times \mathbb{R}_+$, and the symplectizations of h and f are $h^{\mathrm{symp}} = -rp$ and $f^{\mathrm{symp}} = -rz$. Their Hamiltonian vector fields are

$$X_{h^{\mathrm{symp}}} = \frac{\partial}{\partial q}\,, \quad X_{f^{\mathrm{symp}}} = -p\frac{\partial}{\partial p} - z\frac{\partial}{\partial z} + r\frac{\partial}{\partial r}\,.$$

- Consider a section $\chi\colon \mathbb{R}^2 \to M^{\mathrm{symp}}$ of $F^{\mathrm{symp}} = (h^{\mathrm{symp}}, f^{\mathrm{symp}})$ such that $\chi^*\theta = 0$. For instance, one can choose $\chi(\Lambda_1, \Lambda_2) = \left(0, \frac{\Lambda_1}{\Lambda_2}, 1, \Lambda_2\right)$ in the points where $\Lambda_2 \neq 0$.

An example

- The Lie group action $\Phi\colon \mathbb{R}^2 \times M^{\mathrm{symp}} \to M^{\mathrm{symp}}$ defined by the flows of $X_{h^{\mathrm{symp}}}$ and $X_{f^{\mathrm{symp}}}$ is given by

$$\Phi(t, s; q, p, z, r) = \left(q + t, pe^{-s}, ze^{-s}, re^{s}\right) ,$$

whose isotropy subgroup is the trivial one.

- The angle coordinates $(y^0_{\mathrm{symp}}, y^1_{\mathrm{symp}})$ of a point $x \in M^{\mathrm{symp}}$ are determined by

$$\Phi\left(y^0_{\mathrm{symp}}, y^1_{\mathrm{symp}}, \chi(F(x))\right) = x \,.$$

- If the canonical coordinates of x are (q, p, z, r), then

$$y^0_{\mathrm{symp}} = q \,, \quad y^1_{\mathrm{symp}} = -\log z \,.$$

An example

- Since the isotropy subgroup is trivial, the action coordinates coincide with the functions in involution, namely,

$$A_0^{\mathrm{symp}} = h^{\mathrm{symp}} = -rp\,, \quad A_1^{\mathrm{symp}} = f^{\mathrm{symp}} = -rz\,.$$

- Projecting to M yields the functions

$$y^0 = q\,, \quad y^1 = -\log z\,, \quad A_0 = h = p\,, \quad A_1 = f = z\,.$$

An example

- The action coordinate is

$$\tilde{A} = -\frac{A_0}{A_1} = -\frac{p}{z}$$

In the coordinates $(y^0, y^1, \tilde{A})$ the Hamiltonian vector fields reads

$$X_h = \frac{\partial}{\partial y^0}\,, \quad X_f = \frac{\partial}{\partial y^1}\,,$$

and there is a conformal contact form given by

$$\tilde{\eta} = -\frac{1}{A_1}\eta = \mathrm{d}y^1 - \tilde{A}\mathrm{d}y^0\,.$$

An example

- Similarly,
$$\chi(\Lambda_1, \Lambda_2) = \left(\frac{\Lambda_2}{\Lambda_1}, 1, \frac{\Lambda_2}{\Lambda_1}, \Lambda_1\right)$$
is a section of F^{symp} in the points where $\Lambda_1 \neq 0$.
- Performing analogous computations as above one obtains the action-angle coordinates
$$\hat{y}^0 = q - \frac{z}{p}\,, \quad \hat{y}^1 = -\log p\,, \quad \hat{A} = -\frac{z}{p}\,,$$
such that
$$X_h = \frac{\partial}{\partial \hat{y}^0}\,, \quad X_f = \frac{\partial}{\partial \hat{y}^1}\,, \quad \hat{\eta} = -\frac{1}{p}\eta = \mathrm{d}\hat{y}^0 - \hat{A}\mathrm{d}\hat{y}^1\,.$$

$\mathbb{R}^\times$-principal bundles

- Consider the multiplicative group of non-zero real numbers $\mathrm{GL}(1,\mathbb{R}) = \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$.
- Let $\pi \colon P \to M$ be an $\mathbb{R}^\times$-principal bundle, and denote the $\mathbb{R}^\times$-action by Φ, and the Euler vector field by ∇.
- In a local trivialization $\pi^{-1}(U) \simeq U \times \mathbb{R}^\times$ of P, they read

$$\pi(x,s) = x\,, \quad h_t(x,s) = (x, ts)\,, \quad \nabla = s\frac{\partial}{\partial s}\,.$$

Homogeneous symplectic forms

Definition

Let $\pi \colon P \to M$ be an $\mathbb{R}^\times$-principal bundle with Euler vector field ∇. A tensor field A on P is called k-homogeneous (for $k \in \mathbb{Z}$) if

$$\mathcal{L}_\nabla A = kA \,.$$

Definition

A **symplectic $\mathbb{R}^\times$-principal bundle** is an $\mathbb{R}^\times$-principal bundle $\pi \colon P \to M$ endowed with a 1-homogeneous symplectic form ω on P. We will denote it by $(P, \pi, M, \nabla, \omega)$

Contact manifolds and symplectic $\mathbb{R}^\times$-principal bundles

Theorem (Grabowski, 2013)

There is a canonical one-to-one correspondence between contact distributions $C \subset \mathrm{T}M$ on M and symplectic $\mathbb{R}^\times$-principal bundles $\pi\colon P \to M$ over M.

*More precisely, the symplectic $\mathbb{R}^\times$-principal bundle associated with C is $(C^\circ)^\times = C^\circ \setminus 0_{\mathrm{T}^*M} \subset \mathrm{T}^*M$ (i.e., the annihilator of C with the zero section removed), whose symplectic form is the restriction to $(C^\circ)^\times$ of the canonical symplectic form ω_M on T^*Q. It is called the **symplectic cover** of (M, C).*

Remark

Every symplectic $\mathbb{R}^\times$-principal bundle $(P, \pi, M, \nabla, \omega)$ is an exact symplectic manifold. Indeed, the 1-form $\theta = -\iota_\nabla \omega$ is a symplectic potential for ω.
Conversely, an exact symplectic manifold (M, θ) is a symplectic $\mathbb{R}^\times$-principal bundle if the Liouville vector field ∇ is complete.

Contact Hamiltonian vector fields

Theorem (Grabowska and Grabowski, 2022)

Let $(P, \pi, M, \nabla, \omega)$ be the symplectic cover of (M, C). Then, the Hamiltonian vector field X_h of a 1-homogeneous function $h \in \mathscr{C}^\infty(P)$ is π-projectable. The vector field $X_h^c := \mathrm{T}\pi(X_h) \in \mathfrak{X}(M)$ is called the ***contact Hamiltonian vector field*** *of h.*

Proposition

Let $(P^{2n}, \pi, M, \nabla, \omega)$ *be the symplectic cover of the contact manifold* (M, C)*, and let* $F = (f_1, \ldots, f_n)\colon P \to \mathbb{R}^n$ *a map such that* $(M, \theta = -\iota_\nabla \omega, F)$ *is a homogeneous integrable system. Then:*

1. $\pi\left(F^{-1}(\Lambda)\right)$ *is coisotropic, invariant by the flows of* $X^c_{f_1}, \ldots, X^c_{f_n}$*, and diffeomorphic to* $\mathbb{T}^k \times \mathbb{R}^{n-k}$ *for some* $k \leq n$*.*
2. *There exist coordinates* $(y^1, \ldots, y^n, \tilde{A}_1, \ldots, \tilde{A}_{n-1})$ *such that*

$$X^c_{f_\alpha} = \overline{N}^\beta_\alpha \frac{\partial}{\partial y^\beta} ,$$

where $\overline{N}^\beta_\alpha$ *are functions depending only on* $\tilde{A}_1, \ldots, \tilde{A}_n$*.*

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Dziękuję za uwagę!

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