Contact bi-Hamiltonian systems

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Liouville–Arnol'd theorem

Theorem (Liouville–Arnol'd)

Let $f_1, \ldots, f_n$ be independent functions in involution (i.e., $\{f_i, f_j\} = 0\ \forall i, j$) on a symplectic manifold (M^{2n}, ω). Let $M_\Lambda = \{x \in M \mid f_i = \Lambda_i\}$ be a regular level set.

1. *Any compact connected component of M_Λ is diffeomorphic to $\mathbb{T}^n$.*
2. *In a neighborhood of M_Λ there are coordinates (φ^i, J_i) such that*

$$\omega = \mathrm{d}\varphi^i \wedge \mathrm{d}J_i \,,$$

and $f_i = f_i(J_1, \ldots, J_n)$, so the Hamiltonian vector fields read

$$X_{f_i} = \frac{\partial f_i}{\partial J_j} \frac{\partial}{\partial \varphi^j} \,.$$

Liouville–Arnol'd theorem

Corollary

Let (M^{2n}, ω, h) be a Hamiltonian system. Suppose that $f_1, \ldots, f_n$ are independent conserved quantities (i.e. $X_h(f_i) = 0 \ \forall\, i$) in involution. Then, on a neighborhood of M_Λ there are Darboux coordinates (φ^i, J_i) such that $H = H(J_1, \ldots, J_n)$, so the Hamiltonian dynamics are given by

$$\frac{\mathrm{d}\varphi^i}{\mathrm{d}t} = \frac{\partial H}{\partial J_i} \frac{\partial}{\partial \varphi^i}\,,$$

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = 0\,.$$

Problem

Given a Hamiltonian system (M^{2n}, ω, h), we would like to find n independent conserved quantities in involution $f_1, \ldots, f_n$, in order to construct action-angle coordinates (φ^i, J_i).

Magri *et al.* developed a method for constructing such conserved quantities by computing the eigenvalues of a $(1,1)$-tensor field N verifying certain compatibility conditions.

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Magri *et al.* developed a method for constructing such conserved quantities by computing the eigenvalues of a $(1, 1)$-tensor field N verifying certain compatibility conditions.

Compatible Poisson structures

Definition

Let M be a manifold. Two Poisson tensors are Λ and Λ_1 on M are said to be **compatible** if $\Lambda + \Lambda_1$ is also a Poisson tensor on M.

Definition

A vector field $X \in \mathfrak{X}(M)$ is called **bi-Hamiltonian** if it is a Hamiltonian vector field w.r.t. two compatible Poisson structures, namely,

$$X = \Lambda(\mathrm{d}h, \cdot) = \Lambda_1(\mathrm{d}h_1, \cdot)\,,$$

for two functions $h, h_1 \in \mathscr{C}^\infty(M)$.

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Poisson – Nijehuis structures

- The linear map $\sharp_\Lambda \colon \mathrm{T}^*_x M \ni \alpha \mapsto \Lambda(\alpha, \cdot) \in \mathrm{T}_x M$ is an isomorphism iff Λ comes from a symplectic structure ω. In that case, $\sharp_\omega := \sharp_\Lambda^{-1}(v) = \iota_v \omega$.
- In that situation, we can define the $(1,1)$-tensor field

$$N = \sharp_{\Lambda_1} \circ \sharp_\Lambda^{-1}\,.$$

Poisson – Nijehuis structures

Theorem (Magri and Morosi, 1984)

Let (M, ω) *be a symplectic manifold and* Λ_1 *a bivector. Consider the* $(1, 1)$*-tensor field*

$$N = \sharp_{\Lambda_1} \circ \sharp_{\omega}^{-1} .$$

If Λ_1 *is a Poisson tensor compatible with* Λ*, then the Nijehuis torsion* T_N *of* N *vanishes. In that case, the eigenvalues of* N *are in involution w.r.t. both Poisson brackets.*

The pair (Λ, N) *is called a* ***Poisson – Nijenhuis structure*** *on* M*.*

Poisson – Nijehuis structures

Corollary

If a vector field $X \in \mathfrak{X}(M)$ *is bi-Hamiltonian w.r.t. to* ω *and* Λ_1 *(i.e.,* $X = \sharp_\omega \mathrm{d}h = \sharp_{\Lambda_1} \mathrm{d}h_1$*), then the eigenvalues of* N *form a family of conserved quantities in involution w.r.t. both Poisson brackets.*

Proposition (Magri *et al.*, 1997)

Let (Λ, N) *be a Poisson – Nijenhuis structure on* M. *Consider the functions*

$$I_k = \frac{1}{k} \operatorname{Tr} N^k \,, \quad k \in \{1, \ldots, n\} \,.$$

In a neighbourhood of a point $x \in M$ *such that* $\mathrm{d}I_1(x) \wedge \cdots \wedge \mathrm{d}I_n(x) \neq 0$ *there are coordinates* (λ^i, μ_i) *which are canonical both for* Λ *and* N, *namely,*

$$\Lambda = \frac{\partial}{\partial \lambda^i} \wedge \frac{\partial}{\partial \mu_i} \,,$$
$$N^* \mathrm{d}\lambda^i = \lambda^i \mathrm{d}\lambda^i \,,$$
$$N^* \mathrm{d}\mu_i = \lambda^i \mathrm{d}\mu_i \,.$$

Contact geometry

Definition

A (co-oriented) **contact manifold** is a pair (M, η), where M is an $(2n+1)$-dimensional manifold and η is a 1-form on M such that the map

$$\begin{aligned} \flat_\eta \colon \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto \iota_X \mathrm{d}\eta + \eta(X)\eta, \end{aligned}$$

is an isomorphism of $\mathscr{C}^\infty(M)$-modules.

- There exists a unique vector field R on (M, η), called the **Reeb vector field**, given by $R = \flat_\eta^{-1}(\eta)$, or, equivalently,

$$\iota_R \mathrm{d}\eta = 0, \ \ \iota_R \eta = 1.$$

Contact geometry

- The **Hamiltonian vector field** of $f \in \mathscr{C}^\infty(M)$ is given by

$$X_f = \flat_\eta^{-1}(\mathrm{d}f) - (R(f) + f)\, R\,,$$

- Around each point on M there exist **Darboux coordinates** (q^i, p_i, z) such that

$$\eta = \mathrm{d}z - p_i \mathrm{d}q^i,$$
$$R = \frac{\partial}{\partial z},$$
$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + p_i \frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial f}{\partial p_i} - f\right)\frac{\partial}{\partial z}.$$

Contact Hamiltonian systems

Definition

A **contact Hamiltonian system** is a triple (M, η, h) formed by a contact manifold (M, η) and a **Hamiltonian function** $h \in \mathscr{C}^\infty(M)$.

- The dynamics of (M, η, h) is determined by the integral curves of the Hamiltonian vector field X_h of h w.r.t. η.

Contact Hamiltonian systems

- In Darboux coordinates, these curves $c(t) = (q^i(t), p_i(t), z(t))$ are determined by the **contact Hamilton equations**:

$$
\begin{aligned}
\frac{\mathrm{d}q^i(t)}{\mathrm{d}t} &= \frac{\partial h}{\partial p_i} \circ c(t)\,, \\
\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} &= -\frac{\partial h}{\partial q^i} \circ c(t) + p_i(t)\frac{\partial h}{\partial z} \circ c(t)\,, \\
\frac{\mathrm{d}z(t)}{\mathrm{d}t} &= p_i(t)\frac{\partial h}{\partial p_i} \circ c(t) - h \circ c(t)\,.
\end{aligned}
$$

Jacobi manifolds

Definition

A **Jacobi structure** on a manifold M is a pair (Λ, E) where Λ is a bivector and E a vector field such that the composition rule $\{\cdot, \cdot\}$ on $\mathscr{C}^\infty(M)$ given by

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + fE(g) - gE(f)\,,$$

is a Lie bracket, called the **Jacobi bracket**. The triple (M, Λ, E) is called a **Jacobi manifold**.

In particular, $\{\cdot, \cdot\}$ is a Poisson bracket iff $E \equiv 0$.

Jacobi structure of a contact manifold

- A contact manifold (M, η) is endowed with a Jacobi bracket determined by

$$\{f, g\} = -\mathrm{d}\eta(\flat_\eta^{-1}\mathrm{d}f, \flat_\eta^{-1}\mathrm{d}g) - fR(g) + gR(f).$$

- It can also be expressed as follows:

$$\{f, g\} = X_f(g) + gR(f)\,.$$

Jacobi brackets and dissipated quantities

Definition

Let (M, η, h) be a contact Hamiltonian system with Jacobi bracket $\{\cdot, \cdot\}$.
A function $f \in \mathscr{C}^\infty(M)$ is called a **dissipated quantity** if

$$\{f, h\} = 0 \,.$$

Completely integrable contact system

Definition

A **completely integrable contact system** is a triple (M, η, F), where (M, η) is a contact manifold and $F = (f_0, \ldots, f_n)\colon M \to \mathbb{R}^{n+1}$ is a map such that

1. $f_0, \ldots, f_n$ are in involution, i.e., $\{f_\alpha, f_\beta\} = 0\ \forall\, \alpha, \beta$,
2. $\operatorname{rank} \mathrm{T}F \geq n$ on a dense open subset $M_0 \subseteq M$.

Liouville – Arnol'd theorem for contact systems

1. Given $\Lambda \in \mathbb{R}^{n+1} \setminus \{0\}$, let $M_{\langle\Lambda\rangle_+} = \{x \in M \mid \exists\, r \in \mathbb{R}^+ \colon f_\alpha(x) = r\Lambda_\alpha\}$.
2. Assume that the Hamiltonian vector fields $X_{f_0}, \dots, X_{f_n}$ are complete.
3. Let $B \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be an open neighbourhood of Λ.
4. Let $\pi \colon U \to M_{\langle\Lambda\rangle_+}$ be a tubular neighbourhood of $M_{\langle\Lambda\rangle_+}$ such that $F|_U \colon U \to B$ is a trivial bundle over a domain $V \subseteq B$.

Liouville–Arnol'd theorem for contact systems

Theorem (Colombo, de León, Lainz, L.-G., 2023)

Let (M, η, F) be a completely integrable contact system, where $F = (f_0, \ldots, f_n)$. Consider the assumptions of the previous slide. Then:

1. *$M_{\langle\Lambda\rangle_+}$ is coisotropic, invariant by the Hamiltonian flow of f_α, and diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$ for some $k \leq n$.*
2. *There exists coordinates $(y^0, \ldots, y^n, \tilde{A}_1, \ldots, \tilde{A}_n)$ on U such that the equations of motion are given by*

$$\dot{y}^\alpha = \Omega^\alpha(\tilde{A}_1, \ldots, \tilde{A}_n)\,, \quad \dot{\tilde{A}}_i = 0\,.$$

3. *There exists a nowhere-vanishing function $A_0 \in \mathscr{C}^\infty(U)$ and a conformally equivalent contact form $\tilde{\eta} = \eta / A_0$ such that $(y^i, \tilde{A}_i, y^0)$ are Darboux coordinates for $(M, \tilde{\eta})$, namely, $\tilde{\eta} = \mathrm{d}y^0 - \tilde{A}_i \mathrm{d}y^i$.*

Our goal

- We would like to generalize Magri *et al.*'s constructions for integrable contact systems.
- That is, given a contact Hamiltonian system (M, η, h), we want to find a tensor N such that, if it satisfies certain compatibility conditions with (η, h), one can compute dissipated quantities in involution for it.

Compatible Jacobi structures

- Nunes da Costa (1998) introduced the notion of compatibility of Jacobi structures.

Definition

Two Jacobi structures (Λ, E) and (Λ_1, E_1) on a manifold M are said to be **compatible** if $(\Lambda + \Lambda_1, E + E_1)$ is also a Jacobi structure on M.

- She also proved several conditions which are equivalent to (Λ, E) and (Λ_1, E_1) being compatible.

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Jacobi – Nijenhuis structures

- A Jacobi – Nijenhuis structure (Λ, E, N) is a generalization of Nijenhuis – Poisson structures.
- These structures were introduced by Marrero, Monterde and Padrón (1999).
- Their relation with homogeneous Nijenhuis – Poisson structures was studied by Petalidou and Nunes da Costa (2001).

Jacobi – Nijenhuis structures

- The space $\mathfrak{X}(M) \times \mathscr{C}^\infty(M)$ can be endowed with the Lie bracket $[\cdot, \cdot]$ given by
$$\Big[(X, f), (Y, g)\Big] = \Big([X, Y], X(g) - Y(f)\Big)\,.$$
- *Mutatis mutandis*, the Nijenhuis torsion T_N of a linear operator $N\colon \mathfrak{X}(M) \times \mathscr{C}^\infty(M) \to \mathfrak{X}(M) \times \mathscr{C}^\infty(M)$ is defined as usual:
$$\begin{aligned} T_N\Big((X, f), (Y, g)\Big) := \Big[N(X, f), N(Y, g)\Big] - N\Big[N(X, f), (Y, g)\Big] \\ - N\Big[(X, f), N(Y, g)\Big] + N^2\Big[(X, f), (Y, g)\Big]\,, \end{aligned}$$

Jacobi – Nijenhuis structures

- Given a Jacobi structure (Λ, E), we can define the $\mathscr{C}^\infty(M)$-modules homomorphism $\sharp_{(\Lambda,E)} \colon \Omega^1(M) \times \mathscr{C}^\infty(M) \to \mathfrak{X}(M) \times \mathscr{C}^\infty(M)$

$$\sharp_{(\Lambda,E)} \colon (\alpha, f) \mapsto \Big(\Lambda(\cdot, \alpha) + fE, \alpha(E)\Big) .$$

- It is an isomorphism iff (Λ, E) comes from a contact structure.

Jacobi – Nijenhuis structures

Definition

A **Jacobi – Nijenhuis structure** on a manifold M is a triple (Λ, E, N) where (Λ, E) is a Jacobi structure and $N\colon \mathfrak{X}(M) \times \mathscr{C}^\infty(M) \to \mathfrak{X}(M) \times \mathscr{C}^\infty(M)$ is a $\mathscr{C}^\infty(M)$-linear map such that

$$
\begin{aligned}
&N \circ \sharp_{(\Lambda,E)} = \sharp_{(\Lambda,E)} \circ N^* \,,\\
&T_N \equiv 0 \,,\\
&\mathcal{C}\Big((\Lambda, E), N\Big) \equiv 0 \,.
\end{aligned}
$$

The 4-tuple (M, Λ, E, N) is calle a **Jacobi – Nijenhuis manifold**.

Jacobi – Nijenhuis structures

- In the previous slide, $\mathcal{C}$ denotes the **concomitant**. Its expression depends on N, (Λ, E) and a quite involved Lie bracket on $\Omega^1(M) \times \mathscr{C}^\infty(M)$.
- Let (Λ_1, E_1) be the Jacobi structure determined by

$$\Lambda_1(\beta, \alpha) = \Big\langle \beta, N_1\big(\Lambda(\cdot, \alpha), 0\big) \Big\rangle\,, \quad E_1 = N_1(E, 0)\,,$$

  where $N_1 \colon \mathfrak{X}(M) \times \mathscr{C}^\infty(M) \to \mathfrak{X}(M)$ is the projection of N on the first component.
- If (Λ_1, E_1) is also coming from a contact structure, then

$$T_N \equiv 0 \iff \mathcal{C}\Big((\Lambda, E), N\Big) \equiv 0\,.$$

The correspondence between Jacobi–Nijenhuis and homogeneous Nijenhuis–Poisson structures

Proposition (Petalidou and Nunes da Costa, 2001)

With any Jacobi–Nijenhuis manifold (M, Λ, E, N), we can associate a homogeneous Nijenhuis–Poisson manifold, namely, a Nijenhuis–Poisson manifold $(M \times \mathbb{R}, \tilde{\Lambda}, \tilde{N})$ such that

$$\mathcal{L}_{\frac{\partial}{\partial t}} \tilde{\Lambda} = -\tilde{\Lambda}, \quad \mathcal{L}_{\frac{\partial}{\partial t}} \tilde{N} = 0,$$

where t denotes the canonical coordinate on the $\mathbb{R}$ component of $M \times \mathbb{R}$.

Exact symplectic manifolds: Liouville geometry

Definition

An **exact symplectic manifold** is a pair (M, θ), where M is a manifold and θ a one-form on N such that $\omega = -\mathrm{d}\theta$ is a symplectic form on M.

- The **Liouville vector field** Δ of (M, θ) is given by

$$\iota_\Delta \omega = -\theta.$$

- A tensor T is called **homogeneous of degree** n if $\mathcal{L}_\Delta T = nT$.

Symplectization of contact manifolds

Definition

Let (M, η) be a contact manifold and (M^{Σ}, θ) an exact symplectic manifold. A **symplectization** is a fibre bundle $\Sigma\colon M^{\Sigma} \to M$ such that

$$\sigma \Sigma^* \eta = \theta,$$

for a function σ on M^{Σ} called the **conformal factor**.

Symplectization of contact manifolds

Category of contact manifolds
$\updownarrow$
Category of exact symplectic manifolds

- Contact distribution $\ker \eta \longleftrightarrow$ symplectic potential θ
- Functions $\longleftrightarrow$ homogeneous functions
- Hamiltonian vector fields $\longleftrightarrow$ Hamiltonian vector fields

Theorem (Magri's theorem for exact symplectic manifolds)

Let (Λ, N) *be a Poisson – Nijenhuis structure on* M *such that* $\Lambda = \omega^{-1}$ *for an exact symplectic structure* $\omega = -\mathrm{d}\theta$*. Consider the functions*

$$I_k = \frac{1}{k} \operatorname{Tr} N^k \,, \quad k \in \{1, \ldots, n\} \,.$$

In a neighbourhood of a point $x \in M$ *such that* $\mathrm{d}I_1(x) \wedge \cdots \wedge \mathrm{d}I_n(x) \neq 0$ *there are coordinates* (λ^i, μ_i) *which are canonical both for* θ *and* N*, namely,*

$$\theta = \mu_i \mathrm{d}\lambda^i \,,$$
$$N^* \mathrm{d}\lambda^i = \lambda^i \mathrm{d}\lambda^i \,, \quad N^* \mathrm{d}\mu_i = \lambda^i \mathrm{d}\mu_i \,.$$

Moreover, $\mathcal{L}_\Delta \lambda^i = 0$ *and* $\mathcal{L}_\Delta \mu_i = \mu_i$*, where* Δ *is the Liouville vector field w.r.t.* θ*.*

Jacobi – Nijehuis structures $\rightsquigarrow$ action-angle coordinates

- Let (M, η) be a contact manifold with associated Jacobi structure (Λ, E).
- Suppose that there is another contact form η_1 on M with Jacobi structure (Λ_1, E_1).
- Let $N = \sharp_{(\Lambda_1, E_1)} \circ \sharp_{(\Lambda, E)}^{-1}$.
- (Λ, E) and (Λ_1, E_1) are compatible iff $T_N \equiv 0$.
- Let $(M \times \mathbb{R}_+, \theta, \tilde{N})$ denote the symplectization-Poissonization of (M, η, N).
- Let $(\lambda^\alpha, \mu_\alpha)$, $\alpha \in \{1, \ldots, n\}$ be the canonical coordinates adapted to $(\theta, \tilde{N})$.

Jacobi – Nijehuis structures $\rightsquigarrow$ action-angle coordinates

- Unhomogeneizing, we have $2n+2$ functions in M:

$$\overline{\lambda}^{\alpha} = \lambda^{\alpha} \circ \pi_M \,, \quad \overline{\mu}_{\alpha} = \frac{\mu_{\alpha}}{r} \circ \pi_M \,,$$

  where $\pi_M \colon M \times \mathbb{R}_+ \to M$ is the canonical projection and r the global coordinate of $\mathbb{R}_+$.

- We have $(n+1)$ functions in involution w.r.t. the Jacobi bracket:

$$\{\overline{\mu}_{\alpha}, \overline{\mu}_{\beta}\}_{\eta} = 0 \,.$$

- Moreover, they lead to coordinates $(\overline{\lambda}^{\alpha}, \tilde{\mu}_i)$ on M, where $\tilde{\mu}_i = -\frac{\overline{\mu}_i}{\overline{\mu}_j}$ for $i \in \{0, \ldots, n\} \setminus \{j\}$.

Jacobi – Nijehuis structures $\rightsquigarrow$ action-angle coordinates

- In these coordinates,

$$\eta = \mathrm{d}\overline{\lambda}^j - \sum_{i \neq j} \tilde{\mu}_i \mathrm{d}\overline{\lambda}^i \,,$$

$$X_{\overline{\mu}_\alpha} = \frac{\partial}{\partial \overline{\lambda}^\alpha} \,.$$

- Consider a contact Hamiltonian system (M, η, h) such that $X_h = Y_{h_1}$ is the Hamiltonian vector field of h w.r.t. η and the Hamiltonian vector field of h_1 w.r.t. η_1, namely,

$$X_h = Y_{h_1} = \Lambda_1(\cdot, \mathrm{d}h_1) + h_1 E_1 \,.$$

- Then, X_h is given by

$$X_h = f_\alpha(\tilde{\mu}_\beta) \frac{\partial}{\partial \overline{\lambda}^\alpha} \,.$$

Main references

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