Hamilton–Jacobi theory for contact systems: autonomous and non-autonomous

Asier López-Gordón

Instituto de Ciencias Matemáticas (ICMAT-CSIC), Madrid

Joint work with Manuel de León, Manuel Lainz and Xavier Rivas

XXX International Fall Workshop on Geometry and Physics
September 1, 2022

Financially supported by Grants CEX2019-000904-S and PID2019-106715GB-C21 funded by MCIN/AEI/10.13039/501100011033

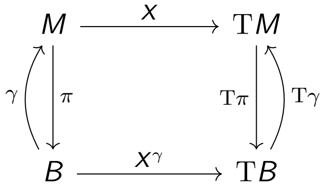
CSIC Consejo Superior de Investigaciones Científicas

ICMAT Instituto de Ciencias Matemáticas

Outline of the presentation

Geometric Hamilton–Jacobi theory

- Consider a dynamical system characterized by $X \in \mathfrak{X}(M)$.
- Suppose that $\pi\colon M \to B$ is a vector bundle (e.g. $\pi_Q\colon \mathrm{T}^*Q \to Q$).
- Idea: obtain a section $\gamma \in \Gamma(M)$ such that the following diagram commutes:



- If σ is an integral curve of X^γ, then $\gamma \circ \sigma$ is an integral curve of X.

Cosymplectic and contact structures

Let M be a $(2n+1)$-dimensional manifold

Cosymplectic manifold (M, ω, τ)

- ω closed 2-form
- τ closed 1-form
- $\tau \wedge \omega^n \neq 0$
- Reeb vector field $\mathcal{R}_t$:

$$\iota_{\mathcal{R}_t}\omega = 0,\ \iota_{\mathcal{R}_t}\tau = 1$$

- Darboux coords. (t, q^i, p_i):

$$\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i,\ \tau = \mathrm{d}t,\ \mathcal{R}_t = \frac{\partial}{\partial t}$$

Contact manifold (M, η)

- η 1-form
- $\eta \wedge \mathrm{d}\eta^n \neq 0$
- Reeb vector field $\mathcal{R}_t$:

$$\iota_{\mathcal{R}_t}\eta = 1, \quad \iota_{\mathcal{R}_t}\mathrm{d}\eta = 0$$

- Darboux coords. (q^i, p_i, z):

$$\eta = \mathrm{d}z - p_i \mathrm{d}q^i,\ \mathcal{R}_z = \frac{\partial}{\partial z}$$

Cocontact structures I

- Idea: a structure that combines the cosymplectic and contact ones.

Definition

A **cocontact manifold** is a triple (M, τ, η) where:

1. M is a $(2n + 2)$-dimensional manifold,
2. τ and η are 1-forms,
3. $\mathrm{d}\tau = 0$,
4. $\tau \wedge \eta \wedge (\mathrm{d}\eta)^{\wedge n} \neq 0$.

Cocontact structures II

- Given a cocontact manifold (M, τ, η), we have the **flat isomorphism**:

$$\begin{aligned} \flat \colon \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto (\iota_X \tau)\tau + \iota_X \mathrm{d}\eta + (\iota_X \eta)\, \eta \end{aligned}$$

  and its inverse $\sharp = \flat^{-1}$.
- **Reeb vector fields**: $\mathcal{R}_t = \flat^{-1}(\tau)$, $\mathcal{R}_z = \flat^{-1}(\eta)$.
- Darboux coordinates $\left(t, q^i, p_i, z\right)$:

$$\tau = \mathrm{d}t, \quad \eta = \mathrm{d}z - p_i \mathrm{d}q^i, \quad \mathcal{R}_t = \frac{\partial}{\partial t}, \quad \mathcal{R}_z = \frac{\partial}{\partial z}$$

Cocontact Hamiltonian systems

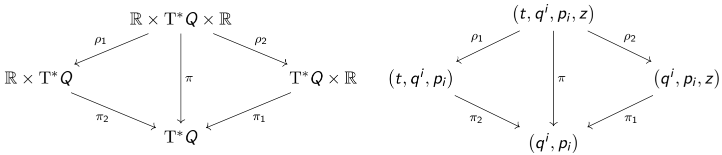
- Given a Hamiltonian function $H\colon M \to \mathbb{R}$, its **Hamiltonian vector field** is given by

$$\flat(X_H) = \mathrm{d}H - (\mathcal{R}_z(H) + H)\,\eta + (1 - \mathcal{R}_t(H))\,\tau\,.$$

- In Darboux coordinates,

$$X_H = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z}\,.$$

Canonical cocontact manifold



- Let Q be an n-dimensional manifold with local coordinates (q^i).
- Let $\theta_0 = p_i \mathrm{d}q^i$ be the canonical 1-form of T^*Q.
- Consider the 1-forms $\theta_Q = \pi^*\theta_0$ and $\eta_Q = \mathrm{d}z - \theta_Q$ on $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$
- Then, $(\mathrm{d}t, \eta_Q)$ is a cocontact structure on $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$. The local expression of the 1-form η is

$$\eta_Q = \mathrm{d}z - p_i \mathrm{d}q^i \,.$$

The action-independent approach

- Let $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \mathrm{d}t, \eta_Q, H)$ be a cocontact Hamiltonian system.
- Idea: obtain a section γ of $\pi_Q^t \colon \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R} \times Q$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} & \xrightarrow{X_H} & \mathrm{T}(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}) \\ \gamma \uparrow \downarrow \pi_Q^t & & \mathrm{T}\pi_Q^t \downarrow \uparrow \mathrm{T}\gamma \\ \mathbb{R} \times Q & \xrightarrow{X_H^\gamma} & \mathrm{T}(\mathbb{R} \times Q) \end{array}$$

- Here $\pi_Q^t \colon (t, q^i, p_i, z) \mapsto (t, q^i)$.

Proposition

$\mathrm{Im}\gamma(t,\cdot)$ *is a Legendrian submanifold* $\forall\ t \in \mathbb{R}$ *(i.e.,* $\gamma^*\eta_Q = 0$*) iff*

$$\gamma(t,q) = j^1_t S(t,q) \coloneqq \left(t, q^i, \frac{\partial S}{\partial q^i}, S\right)$$

Theorem (Action-independent Hamilton–Jacobi Theorem)

Suppose that, $\forall t \in \mathbb{R}$, $\mathrm{Im}\gamma(t,\cdot)$ *is a Legendrian submanifold. Then,* X_H^γ *and* X_H *are* γ*-related iff*

$$H \circ j^1_t S + \frac{\partial S}{\partial t} = 0\,.$$

The function S *is called a* ***generating function*** *for* H*.*

Application: time-independent contact systems

- Let $(\mathrm{T}^*Q \times \mathbb{R}, \eta_Q, H)$ be a contact Hamiltonian system.
- Consider the associated cocontact Hamiltonian system $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \mathrm{d}t, \eta_Q, H \circ \rho_2)$, where $\rho_2 \colon (t, q^i, p_i, z) \mapsto (q^i, p_i, z)$.
- Suppose that $S(t, q) = \alpha(q) + \beta(t)$.
- Then, the Hamilton–Jacobi equation is written as

$$H \circ j^1\alpha + \frac{\partial \beta}{\partial t} = 0\,,$$

- Notice that if S is time-independent (i.e., $\beta = 0$), the solutions of the HJ problem only cover the zero energy level!

Example (The free particle with linear dissipation)

- Consider the contact Hamiltonian system $(\mathrm{T}^*\mathbb{R} \times \mathbb{R}, \eta_{\mathbb{R}}, H)$ with

$$H(q, p, z) = \frac{p^2}{2} + \delta z\,.$$

- Then, $S(t, q) = \lambda e^{-\delta t} - \frac{\delta}{2} q^2$ is a generating function for H.
- Now,

$$X_H^\gamma = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}\bigg|_{\operatorname{Im}\gamma} = \frac{\partial}{\partial t} - \delta q \frac{\partial}{\partial q},$$

whose integral curves are of the form $\sigma(t) = (t, q_0 e^{-\delta t})$.
- Thus, the integral curves of $X_{H|\operatorname{Im}(\gamma)}$ are given by

$$\gamma \circ \sigma(t) = (t, q_0 e^{-\delta t}, -\delta q_0 e^{-\delta t}, -\frac{\delta}{2} q_0^2 e^{-2\delta t} + \lambda e^{-\delta t}).$$

The action-dependent approach

- The previous approach has a drawback: complete solutions cannot be defined.
- Idea: consider sections of $\pi_Q^{t,z} \colon \mathbb{R} \times \mathrm{T}^* Q \times \mathbb{R} \to \mathbb{R} \times Q \times \mathbb{R}$:

$$
\begin{array}{ccc}
\mathbb{R} \times \mathrm{T}^* Q \times \mathbb{R} & \xrightarrow{\quad X_H \quad} & \mathrm{T}(\mathbb{R} \times \mathrm{T}^* Q \times \mathbb{R}) \\
\gamma \uparrow \downarrow \pi_Q^{t,z} & & \mathrm{T}\pi_Q^{t,z} \downarrow \uparrow \mathrm{T}\gamma \\
\mathbb{R} \times Q \times \mathbb{R} & \xrightarrow{\quad X_H^\gamma \quad} & \mathrm{T}(\mathbb{R} \times Q \times \mathbb{R})
\end{array}
$$

Proposition

If (M, τ, η) is a cocontact manifold, then $(M, \Lambda, -\mathcal{R}_z)$ is a Jacobi manifold, where $\Lambda(\alpha, \beta) = -\mathrm{d}\eta(\sharp\alpha, \sharp\beta)$.

- Recall that the orthogonal complement $\mathcal{D}^\perp$ of a distribution $\mathcal{D} \subseteq \mathrm{T}M$ is given by $\mathcal{D}^\perp = \Lambda(\mathcal{D}^\circ, \cdot)$.
- A submanifold N is said to be **coisotropic** if $\mathrm{T}N^\perp \subseteq \mathrm{T}N$.

- Let $\mathrm{d}_Q f := \frac{\partial f}{\partial q^i}\mathrm{d}q^i$ for $f \in C^\infty(\mathbb{R} \times Q \times \mathbb{R})$.

Theorem (Action-dependent Hamilton–Jacobi Theorem)

Let γ be a section of $\pi_Q^{t,s} : \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \to \mathbb{R} \times Q \times \mathbb{R}$ such that $\operatorname{Im}\gamma$ is a coisotropic submanifold. Then, X_H^γ and X_H are γ-related iff

$$\mathrm{d}_Q\left(H \circ \gamma\right) + \mathcal{L}_{\mathcal{R}_z}(H \circ \gamma)\,\gamma + \mathcal{L}_{\mathcal{R}_t}\gamma = \left(H \circ \gamma\right)\,\mathcal{L}_{\mathcal{R}_z}\gamma\,.$$

Complete solutions I

Definition

A **complete solution of the Hamilton–Jacobi problem** is a local diffeomorphism $\Phi\colon \mathbb{R} \times Q \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$ such that, for each $\lambda \in \mathbb{R}^n$,

$$\begin{aligned}\Phi_\lambda\colon \mathbb{R} \times Q \times \mathbb{R} &\longrightarrow \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}\\ \left(t, q^i, z\right) &\longmapsto \Phi\left(t, q^i, \lambda, z\right)\end{aligned}$$

is a solution of the action-dependent HJ problem.

Complete solutions II

- Let us define the functions $f_i = \pi_i \circ \alpha \circ \Phi^{-1}$ on $\mathbb{R} \times \mathrm{T}^* Q \times \mathbb{R}$, so that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{R} \times Q \times \mathbb{R} \times \mathbb{R}^n & \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} & \mathbb{R} \times \mathrm{T}^* Q \times \mathbb{R} \\
\Big\downarrow \alpha & & \Big\downarrow f_i \\
\mathbb{R}^n & \xrightarrow{\quad \pi_i \quad} & \mathbb{R}
\end{array}
$$

Theorem

For each $i \in \{1, \ldots, n\}$, the function $f_i = \pi_i \circ \alpha \circ \Phi^{-1}$ is a constant of the motion. However, these functions are not necessarily in involution, i.e., $\{f_i, f_j\} \neq 0$.

Freely falling particle with linear dissipation I

- The Hamiltonian function $H : \mathbb{R} \times \mathrm{T}^*\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$H(t, q, p, z) = \frac{p^2}{2m(t)} + m(t)gq + \frac{\gamma}{m(t)}z \,.$$

- We look for a conserved quantity f, i.e., $X_H(f) = 0$.
- For simplicity's sake, suppose that $f = f(t, p)$.
- Conserved quantity:

$$f(t, q, p, z) = e^{\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s} \left(p + g e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s} \int_1^t e^{-\int_1^u -\frac{\gamma}{m(s)}\mathrm{d}s} m(u)\mathrm{d}u \right) \,.$$

Freely falling particle with linear dissipation II

- We can thus express the momentum p as a function of t and the real parameter λ, namely,

$$P(t,\lambda) = e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\left(\lambda - g e^{-\int_1^t \frac{\gamma}{m(s)}\mathrm{d}s}\int_1^t e^{\int_1^u \frac{\gamma}{m(s)}\mathrm{d}s} m(u)\mathrm{d}u\right) ,$$

and obtain a complete solution of the Hamilton–Jacobi problem for H:

$$\phi_\lambda : (t,q,z) \longmapsto (t,q,p \equiv P(t,\lambda), z) .$$

Damped forced harmonic oscillator I

- Consider the product manifold $\mathbb{R} \times \mathrm{T}Q \times \mathbb{R}$ with Hamiltonian function

$$H(t, q, p, s) = \frac{p^2}{2m} + \frac{k}{2} q^2 - qF(t) + \frac{\gamma}{m} s \,.$$

- Conserved quantity:

$$g(t, q, p) = e^{\frac{\gamma t}{2m}} \left(\frac{\sinh\left(\frac{\kappa t}{2m}\right)(2kmq + \gamma p)}{\kappa} + p \cosh\left(\frac{\kappa t}{2m}\right) \right) - \int_1^t F(s) e^{\frac{\gamma s}{2m}} \left(\cosh\left(\frac{\kappa s}{2m}\right) + \frac{\gamma \sinh\left(\frac{\kappa s}{2m}\right)}{\kappa} \right) \mathrm{d}s \,,$$

where $\kappa = \sqrt{\gamma^2 - 4km}$.

Damped forced harmonic oscillator II

- Thus, we can write p in terms of t, q and a real parameter λ as

$$P(t,q,\lambda) = \frac{e^{-\frac{\gamma t}{2m}}}{\gamma \sinh\left(\frac{\kappa t}{2m}\right) + \kappa \cosh\left(\frac{\kappa t}{2m}\right)} \left[\kappa\lambda - 2kmqe^{\frac{\gamma t}{2m}} \sinh\left(\frac{\kappa t}{2m}\right)\right.$$
$$\left. \kappa \int_1^t e^{\frac{s\gamma}{2m}} F(s) \left(\cosh\left(\frac{\kappa s}{2m}\right) + \frac{\gamma \sinh\left(\frac{\kappa s}{2m}\right)}{\kappa}\right) \mathrm{d}s\right],$$

and obtain a complete solution of the Hamilton–Jacobi problem:

$$\Phi_\lambda \colon (t,q,\lambda,z) \mapsto (t,q,p \equiv P(t,q,\lambda), z) .$$

[1] M. de León, J. Gaset, X. Gràcia, M. C. Muñoz-Lecanda, and X. Rivas, *Time-dependent contact mechanics*, to appear in Monatsh. Math., May 19, 2022. arXiv: 2205.09454 [math-ph].

[2] M. de León, M. Lainz, A. L.-G., and X. Rivas, *Hamilton–Jacobi theory for contact systems: Autonomous and non-autonomous*, Aug. 15, 2022. arXiv: 2208.07436 [math-ph].

Thank you!

✉ asier.lopez@icmat.es

🌐 www.alopezgordon.xyz