Mechanical systems with external forces

Symmetries, reduction and Hamilton-Jacobi theory

Asier López Gordón

Instituto de Ciencias Matemáticas (ICMAT-CSIC)

Joint work with Manuel de León and Manuel Lainz

XXIX International Fall Workshop in Geometry and Physics
September 8, 2021

Symplectic structure on TQ induced by the Lagrangian I

- A **symplectic manifold** (M, ω) is an $2m$-dimensional manifold M endowed with a symplectic form ω (i.e., a closed and non-degenerate 2-form).
- The **vertical endomorphism** $S : T(TQ) \to T(TQ)$ is given by

$$S\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial \dot{q}^i}, \qquad S\left(\frac{\partial}{\partial \dot{q}^i}\right) = 0.$$

- Its adjoint $S^* : T^*(TQ) \to T^*(TQ)$ is given by

$$S^*(\mathrm{d}q^i) = 0, \qquad S^*(\mathrm{d}\dot{q}^i) = \mathrm{d}q^i.$$

Symplectic structure on TQ induced by the Lagrangian II

- Consider a Lagrangian function L on TQ.
- The Poincaré-Cartan forms are given by

$$\theta_L = S^*(\mathrm{d}L), \qquad \omega_L = -\mathrm{d}\theta_L.$$

- Hereinafter, L will be assumed to be regular, i.e., ω_L is symplectic.
- The Liouville vector field Δ on TQ is given by

$$\Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}.$$

SODE

- A **second order differential equation** (**SODE**) is a vector field ξ on TQ that is a section of both $\tau_{TQ} : TTQ \to TQ$ and $T\tau_Q : TTQ \to TQ$.
- Locally,

$$\xi = \dot{q}^i \frac{\partial}{\partial q^i} + \xi^i(q^i, \dot{q}^i) \frac{\partial}{\partial \dot{q}^i}.$$

- Clearly, ξ is a SODE if and only if

$$S(\xi) = \Delta.$$

- A **solution** of a SODE ξ is a curve $\sigma(t) = (q^i(t))$ on Q such that its canonical lift to TQ is an integral curve of ξ, given by

$$\frac{\mathrm{d}^2 q^i}{\mathrm{d}t^2} = \xi^i \left(q^i, \frac{\mathrm{d}q^i}{\mathrm{d}t} \right), \quad 1 \leq i \leq n = \dim Q.$$

Forced Euler-Lagrange equations

- An external force is represented by a semibasic 1-form α on TQ. Locally,
$$\alpha = \alpha_i(q, \dot{q})\, \mathrm{d}q^i.$$
- The dynamics is determined by the **forced Euler-Lagrange vector field** $\xi_{L,\alpha}$, given by
$$\iota_{\xi_{L,\alpha}} \omega_L = \mathrm{d}E_L + \alpha,$$
where $E_L = \Delta(L) - L$.
- $\xi_{L,\alpha}$ is a SODE, with solutions given by
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = -\alpha_i, \quad 1 \leq i \leq n.$$

Vertical and complete lifts of a vector field

- Consider a vector field X on Q locally given by

$$X = X^i \frac{\partial}{\partial q^i}.$$

- Its **vertical lift** is the vector field X^v on TQ given by

$$X^v = X^i \frac{\partial}{\partial \dot{q}^i}.$$

- Its **complete lift** is the vector field X^c on TQ given by

$$X^c = X^i \frac{\partial}{\partial q^i} + \dot{q}^j \frac{\partial X^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i}.$$

Rayleigh forces

- An **Rayleigh force** is an external force of the form

$$\bar{R} = S^*(\mathrm{d}\mathcal{R}),$$

where $\mathcal{R} : TQ \to \mathbb{R}$ is the **Rayleigh potential** or **Rayleigh dissipation function**.

- $\mathcal{R}$ expresses the energy dissipated away by the system:

$$\frac{\mathrm{d}}{\mathrm{d}t} E_L \circ \sigma(t) = -\Delta(\mathcal{R}) \circ \sigma(t),$$

with σ an integral curve of $\xi_{L,\bar{R}}$.

Dissipative bracket

- The **dissipative bracket** of a pair of functions f and g on (TQ, ω_L) is given by

$$[f, g] := (SX_f)(g) = \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)^{-1} \frac{\partial f}{\partial \dot{q}^i} \frac{\partial g}{\partial \dot{q}^j}$$

- It is bilinear and symmetric
- It satisfies the Leibniz rule:

$$[fg, h] = [f, h]g + f[g, h]$$

- f is a constant of the motion of $(L, \mathcal{R})$ iff

$$\{f, E_L\} - [f, \mathcal{R}] = 0.$$

Noether theorem

Theorem (Noether's theorem for forced Lagrangian systems)

Let X be a vector field on Q. Then $X^c(L) = \alpha(X^c)$ if and only if $X^v(L)$ is a constant of the motion.

- A vector field X on Q satisfying these conditions is called a **symmetry of the forced Lagrangian** (L, α).
- For a Rayleigh system $(L, \mathcal{R})$, this is equivalent to

$$X^c(L) = X^v(\mathcal{R}).$$

Example (Fluid resistance)

- Consider a body of mass m moving along 1 dimension through a fluid that fully encloses it.
- The Rayleigh potential associated to the drag force is

$$\mathcal{R} = \frac{k}{3}\dot{q}^3, \qquad k = \frac{1}{2}CA\rho; \qquad L = \frac{1}{2}m\dot{q}^2.$$

- Consider the vector field

$$X = e^{kq/m}\frac{\partial}{\partial q}.$$

- $X^c(L) = X^v(\mathcal{R}) \Longrightarrow X^v(L) = me^{kq/m}\dot{q}$ is a constant of the motion.
- When $k \to 0$, X is the generator of translations and the conservation of momentum is recovered.

Other point-like symmetries I

- A **Lie symmetry** is a vector field X on Q such that

$$[X^c, \xi_{L,\alpha}] = \mathcal{L}_{X^c}\xi_{L,\alpha} = 0$$

- If $\mathcal{L}_{X^c}\alpha_L$ is closed, then X is a Lie symmetry if and only if

$$\mathcal{L}_{X^c}\alpha = -\mathrm{d}(X^c(E_L)).$$

- A **Noether symmetry** is a vector field X on Q such that

$$\mathcal{L}_{X^c}\alpha_L = \mathrm{d}f, \qquad X^c(E_L) + \alpha(X^c) = 0.$$

- If $\mathcal{L}_{X^c}\alpha_L = \mathrm{d}f$, then X is a Noether symmetry if and only if $f - X^v(L)$ is a conserved quantity.

Other point-like symmetries II

- For a Rayleigh system $(L, \mathcal{R})$, if $\mathcal{L}_{X^c}\alpha_L = \mathrm{d}f$, then X is a Noether symmetry if and only if

$$X^c(E_L) + X^v(\mathcal{R}) = 0.$$

- If X is a Noether symmetry, it is also a symmetry of the forced Lagrangian if and only if $\mathcal{L}_{X^c}\alpha_L = 0$.
- If X is a Noether symmetry, it is also a Lie symmetry if and only if

$$\iota_{X^c}\mathrm{d}\alpha = 0.$$

Non-point-like symmetries I

- A vector field $\tilde{X}$ on TQ is called a **dynamical symmetry** if

$$[\tilde{X}, \xi_{L,\alpha}] = 0.$$

- A vector field $\tilde{X}$ on TQ is called a **Cartan symmetry** if

$$\mathcal{L}_{\tilde{X}}\alpha_L = \mathrm{d}f, \qquad \tilde{X}(E_L) + \alpha(\tilde{X}) = 0$$

- X is a Lie symmetry if and only if X^c is a dynamical symmetry.
- X is a Noether symmetry if and only if X^c is a Cartan symmetry.

Non-point-like symmetries II

- If $\mathcal{L}_{\tilde{X}}\alpha_L$ is closed, then $\tilde{X}$ is a dynamical symmetry if and only if

$$\mathrm{d}(\tilde{X}(E_L)) = -\mathcal{L}_{\tilde{X}}\alpha.$$

- A Cartan symmetry is a dynamical symmetry if and only if

$$\iota_{\tilde{X}}\mathrm{d}\alpha = 0.$$

- If $\mathcal{L}_{\tilde{X}}\alpha_L = \mathrm{d}f$, then $\tilde{X}$ is a Cartan symmetry if and only if $f - (S\tilde{X})(L)$ is a constant of the motion.
- For a Rayleigh system $(L, \mathcal{R})$, $\tilde{X}$ is a Cartan symmetry if and only if

$$\tilde{X}(E_L) + (S\tilde{X})(\mathcal{R}) = 0.$$

Momentum map

- Consider a G-invariant regular Lagrangian L on TQ, where G is a Lie group with Lie algebra $\mathfrak{g}$ and dual Lie algebra $\mathfrak{g}^*$.
- Assume the G-action to be free and proper.
- The **natural momentum map** is given by

$$\begin{aligned} J : TQ &\to \mathfrak{g}^* \\ \langle J(x), \xi \rangle &= \theta_L(\xi_Q^c) \end{aligned}$$

  for each $\xi \in \mathfrak{g}$.
- For each $\xi \in \mathfrak{g}$, we can introduce a function on TQ:

$$\begin{aligned} J^\xi : TQ &\to \mathbb{R} \\ x &\mapsto \langle J(x), \xi \rangle \end{aligned}$$

Lemma

Consider a forced Lagrangian system (L, α)*. Let* $\xi \in \mathfrak{g}$*. Then*

1. J^ξ *is a conserved quantity if and only if*

$$\alpha(\xi_Q^c) = 0.$$

2. *If the previous equation holds, then* ξ *leaves* α *invariant if and only if*

$$\iota_{\xi_Q^c} \mathrm{d}\alpha = 0.$$

In addition, the vector subspace of $\mathfrak{g}$ *given by*

$$\mathfrak{g}_\alpha = \left\{ \xi \in \mathfrak{g} \mid \alpha(\xi_Q^c) = 0,\ \iota_{\xi_Q^c} \mathrm{d}\alpha = 0 \right\}$$

is a Lie subalgebra of $\mathfrak{g}$*.*

Isotropy group

- J^{ξ} is a constant of the motion $\forall \xi \in \mathfrak{g}_\alpha \Rightarrow J_\alpha^{-1}(\mu)$ is left invariant by the flow of $\xi_{L,\alpha}$.
- Therefore the integral curves of $\xi_{L,\alpha}$ are contained in level sets $J_\alpha^{-1}(\mu) \subset TQ$.
- The isotropy Lie algebra at $\mu \in \mathfrak{g}_\alpha^*$ is

$$(\mathfrak{g}_\alpha)_\mu = \{\xi \in \mathfrak{g}_\alpha \mid \langle \mu, [\xi, \eta] \rangle = 0 \ \forall \eta \in \mathfrak{g}_\alpha\} .$$

- $(G_\alpha)_\mu \leq G$ is the Lie group generated by $(\mathfrak{g}_\alpha)_\mu$.

Adjoint and coadjoint actions and isotropy group

For $g \in G_\alpha,\ \xi \in \mathfrak{g}_\alpha,\ \mu \in \mathfrak{g}_\alpha^*$:

- The adjoint action is given by

$$\mathrm{Ad}_g \xi = \left.\frac{\mathrm{d}}{\mathrm{d}t} g \exp(t\xi) g^{-1}\right|_{t=0}.$$

- The coadjoint action is given by

$$\left\langle \mathrm{Ad}_g^* \mu, \xi \right\rangle = \langle \mu, \mathrm{Ad}_g \xi \rangle$$

- The isotropy group for $\mu \in \mathfrak{g}_\alpha^*$ is

$$(G_\alpha)_\mu = \left\{ g \in G_\alpha \mid \mathrm{Ad}_g^* \mu = \mu \right\}$$

Theorem

Consider a $\mathfrak{g}_\alpha$-invariant forced Lagrangian system (L, α) on TQ. Let $\mu \in \mathfrak{g}_\alpha^*$. Then:

1. *The quotient space $(TQ)_\mu := J_\alpha^{-1}(\mu)/(G_\alpha)_\mu$ is endowed with an induced symplectic structure ω_μ, given by*

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega_L,$$

where $\pi_\mu : J_\alpha^{-1}(\mu) \to (TQ)_\mu$ and $i_\mu : J_\alpha^{-1}(\mu) \hookrightarrow TQ$.

2. *The reduced Lagrangian L_μ is given by*

$$L_\mu \circ \pi_\mu = L \circ i_\mu.$$

3. *The reduced external force α_μ is given by*

$$\pi_\mu^* \alpha_\mu = i_\mu^* \alpha.$$

Standard Hamilton-Jacobi problem

- The Hamilton-Jacobi problem consists in finding a characteristic function W on Q such that

$$H\left(q^i, \frac{\partial W}{\partial q^i}\right) = E.$$

- Geometrically, this equation can be written as

$$\gamma^* H = E,$$

with $\gamma = \mathrm{d}W$ a section of T^*Q.

Hamilton-Jacobi problem for (H, β)

Theorem

Let γ be a closed 1-form on Q. Then the following conditions are equivalent:

1. $\mathrm{d}(H \circ \gamma) = -\gamma^* \beta$,
2. *for every curve $\sigma : \mathbb{R} \to Q$ such that*
$$\dot{\sigma}(t) = T\pi_Q \circ X_{H,\beta} \circ \gamma \circ \sigma(t)$$
for all t, then $\gamma \circ \sigma$ is an integral curve of $X_{H,\beta}$;
3. $\operatorname{Im} \gamma$ *is a Lagrangian submanifold of T^*Q and $X_{H,\beta}$ is tangent to it.*

If γ satisfies these conditions, it is called a solution of the Hamilton-Jacobi problem for (H, β),

Complete solutions I

- A map $\Phi : Q \times \mathbb{R}^n \to T^*Q$ is called **complete solution of the Hamilton-Jacobi problem** for (H, β) if
  1. Φ is a local diffeomorphism,
  2. for any $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, the map

$$\begin{aligned} \Phi_\lambda : Q &\to T^*Q \\ q &\mapsto \Phi_\lambda(q) = \Phi(q, \lambda_1, \ldots, \lambda_n) \end{aligned}$$

  is a solution of the Hamilton-Jacobi problem for (H, β).
- Assume Φ to be a global diffeomorphism.

Complete solutions II

- Consider the functions given by

$$f_a = \pi_a \circ \Phi^{-1} : T^*Q \to \mathbb{R},$$

where π_a denotes the projection over the a-th component of $\mathbb{R}^n$.

- The functions f_a are constants of the motion. Moreover, they are in involution, i.e.,

$$\{f_a, f_b\} = 0$$

Example

Consider a n-dimensional forced Hamiltonian system (H, β), with

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2, \qquad \beta = \sum_{i=1}^{n} \kappa_i p_i^2 \mathrm{d}q_i.$$

The functions

$$f_a = e^{\kappa_a q^a} p_a, \qquad a = 1, \ldots, n.$$

are constants of the motion in involution. The 1-form γ on Q given by

$$\gamma = \sum_{i=1}^{n} \lambda_i e^{-\kappa_i q^i} \mathrm{d}q^i$$

is a complete solution of the Hamilton-Jacobi problem.

Discrete mechanics I

- The continuous objects are now replaced by their discrete counterparts: $TQ \rightsquigarrow Q \times Q$
- The **exact** discrete Lagrangian and external forces are

$$L_d^{\mathrm{ex}}(q_j, q_{j+1}) = \int_{t_j}^{t_{j+1}} L(q(t), \dot{q}(t))\, \mathrm{d}t,$$

$$f_d^{E+}(q_j, q_{j+1}) = -\int_{t_j}^{t_{j+1}} \alpha(q(t), \dot{q}(t)) \cdot \frac{\partial q(t)}{\partial q_{j+1}}\, \mathrm{d}t,$$

$$f_d^{E-}(q_j, q_{j+1}) = -\int_{t_j}^{t_{j+1}} \alpha(q(t), \dot{q}(t)) \cdot \frac{\partial q(t)}{\partial q_j}\, \mathrm{d}t.$$

- In practice, one takes an approximation of the integrals above, e.g., by the trapezoidal rule.

Discrete mechanics II

- The **forced discrete Legendre transforms** define the following momenta:

$$p_{j+1} = D_2 L_d (q_j, q_{j+1}) + f_d^+ (q_j, q_{j+1}),$$
$$p_j = -D_1 L_d (q_j, q_{j+1}) - f_d^- (q_j, q_{j+1}).$$

- We can define the **right discrete Hamiltonian**:

$$H_d^+ (q_j, p_{j+1}) = p_{j+1} q_{j+1} - L_d (q_j, q_{j+1}),$$

- The **discrete action** is

$$S_d^N (\{q_j\}) = \sum_{j=0}^{N-1} L_d (q_j, q_{j+1}) = \sum_{j=0}^{N-1} \left[p_{j+1} q_{j+1} - H_d^+ (q_j, p_{j+1}) \right]$$

Discrete mechanics III

- The dynamics is given by the **discrete Lagrange-d'Alembert principle**:

$$\delta S_d^N(\{q_j\}) + \sum_{k=0}^{N-1} \left[f_d^-(q_k, q_{k+1})\, \delta q_k + f_d^+(q_k, q_{k+1})\, \delta q_{k+1} \right] = 0$$

- From the discrete Lagrange-d'Alembert principle, one can derive the **forced right discrete Hamilton equations**:

$$\left[q_{j+1} - D_2 H_d^+(q_j, p_{j+1}) \right] \frac{\partial p_{j+1}}{\partial q_{j+1}} = -f_d^+(q_j, q_{j+1}),$$
$$p_j = D_1 H_d^+(q_j, p_{j+1}) - f_d^-(q_j, q_{j+1}).$$

Forced discrete Hamilton-Jacobi theory I

- Let us introduce the following mappings

$$
\begin{aligned}
\gamma^{+} \coloneqq DS_d \circ \pi_2 + f_d^{+} : Q \times Q &\to T^*Q \\
(q_j, q_{j+1}) &\mapsto (q_{j+1}, p_{j+1}),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{F}^{+} : Q \times Q \to Q \\
&\mathcal{F}^{+}(q_{j-1}, q_j) \coloneqq \; D_2 H_d^{+}\left(q_j, \gamma^{+}(q_j, \mathcal{F}^{+}(q_{j-1}, q_j))\right) \\
&\qquad - f_d^{+}\left(q_j, \mathcal{F}^{+}(q_{j-1}, q_j)\right) \left[D_2\gamma^{+}(q_j, \mathcal{F}^{+}(q_{j-1}, q_j))\right]^{-1}
\end{aligned}
$$

Forced discrete Hamilton-Jacobi theory II

Theorem

Suppose that

1. S_d *and* γ^+ *satisfy the* ***forced right discrete H-J equation****:*

$$S_d^{j+1}(q_{j+1}) - S_d^j(q_j) - \gamma^+(q_j, q_{j+1})q_{j+1} + H_d^+(q_j, \gamma^+(q_j, q_{j+1})) = 0,$$

2. *the sequence of points* $\{c_k\}_{k=0}^N \subset Q$ *satisfies*

$$c_{k+1} = \mathcal{F}^+(c_{k-1}, c_k).$$

Then, the set of points $\{(c_k, p_k)\}_{k=0}^N \subset T^*Q$ *with*

$$p_{k+1} = \gamma^+(q_{k-1}, q_k)$$

is a solution of the forced right discrete Hamilton equations.

Main references I

- J. F. Cariñena, X. Gràcia, G. Marmo, *et al.*, "Geometric Hamilton-Jacobi theory," *Int. J. Geom. Methods Mod. Phys.*, vol. 03, no. 07, pp. 1417–1458, 2006.
- D. Iglesias-Ponte, M. de León, and D. M. de Diego, "Towards a Hamilton-Jacobi theory for nonholonomic mechanical systems," *J. Phys. A: Math. Theor.*, vol. 41, no. 1, p. 015 205, Jan. 11, 2008.
- M. de León, M. Lainz, and A. López-Gordón, *Geometric Hamilton-Jacobi theory for systems with external forces and discretization*, Under preparation, 2021.
- ——,"Symmetries, constants of the motion, and reduction of mechanical systems with external forces," *J. Math. Phys.*, vol. 62, no. 4, p. 042 901, 2021. arXiv: 2101.09036.

Main references II

- M. de León and C. Sardón, "Geometry of the discrete Hamilton-Jacobi equation: Applications in optimal control," *Rep. Math. Phys.*, vol. 81, no. 1, pp. 39–63, Feb. 1, 2018.
- A. López-Gordón, "The geometry of Rayleigh dissipation," MSc thesis, Universidad Autónoma de Madrid, 2021. arXiv: 2107.03780.
- J. E. Marsden and M. West, "Discrete mechanics and variational integrators," *Acta Numer.*, vol. 10, pp. 357–514, May 2001.