Non-conservative systems can have conserved quantities!

Symmetries, reduction and Hamilton-Jacobi theory for forced mechanical systems

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34th International Colloquium on
Group Theoretical Methods in Physics
July 20, 2022

Financially supported by Grants CEX2019-000904-S and PID2019-106715GB-C21 funded by MCIN/AEI/10.13039/501100011033

Outline of the presentation

Motivation

External forces appear in many dynamical systems:

- systems with dissipation or friction,
- control forces,
- nonholonomic Čaplygin systems.

Geometric Lagrangian mechanics

- Let Q be an n-manifold with coords. (q^i): the space of positions.
- TQ, with coords. $(q^i, \dot{q}^i)$, is the space of positions and velocities.
- Consider a Lagrangian function $L\colon TQ \to \mathbb{R}$.
- The Poincaré-Cartan forms are given by

$$\theta_L = \frac{\partial L}{\partial \dot{q}^i} \mathrm{d} q^i, \qquad \omega_L = -\mathrm{d}\theta_L .$$

- Hereinafter, L will be assumed to be regular, i.e.,

$$\det \left(\frac{\partial^2 L}{\partial \dot{q}^i \dot{q}^j} \right) \neq 0 \,.$$

Forced Euler-Lagrange equations

- An external force is represented by a semibasic 1-form $\alpha = \alpha_i(q, \dot{q})\, \mathrm{d}q^i$.
- The dynamics is determined by the **forced Euler-Lagrange vector field** $\Gamma_{L,\alpha}$, given by

$$\omega_L(\Gamma_{L,\alpha},\ \cdot) = \mathrm{d}E_L + \alpha,$$

  where $E_L = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L$.
- The integral curves of $\Gamma_{L,\alpha}$ satisfy the **forced Euler-Lagrange equations**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = -\alpha_i, \quad 1 \leq i \leq n.$$

Rayleigh forces

- An **Rayleigh force** is an external force of the form

$$\alpha = \frac{\partial \mathcal{R}}{\partial \dot{q}^i} \, \mathrm{d}q^i,$$

where $\mathcal{R} : TQ \to \mathbb{R}$ is the **Rayleigh potential** or **Rayleigh dissipation function**.

- $\mathcal{R}$ expresses the energy dissipated away by the system:

$$\frac{\mathrm{d}E_L}{\mathrm{d}t} = -\dot{q}^i \frac{\partial \mathcal{R}}{\partial \dot{q}^i} \, .$$

- Rayleigh considered only forces linear in the velocities, namely,

$$\mathcal{R} = \frac{1}{2} R_{ij}(q) \dot{q}^i \dot{q}^j \, .$$

Vertical and complete lifts of a vector field

- Consider a vector field X on Q locally given by

$$X = X^i \frac{\partial}{\partial q^i}.$$

- Its **vertical lift** is the vector field X^v on TQ given by

$$X^v = X^i \frac{\partial}{\partial \dot{q}^i}.$$

- Its **complete lift** is the vector field X^c on TQ given by

$$X^c = X^i \frac{\partial}{\partial q^i} + \dot{q}^j \frac{\partial X^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i}.$$

Noether theorem

- A vector field X on Q is called a **symmetry of the forced Lagrangian** (L, α) if $X^c(L) = \alpha(X^c)$.
- For a Rayleigh system $(L, \mathcal{R})$, this is equivalent to

$$X^c(L) = X^v(\mathcal{R}).$$

Theorem (Noether's theorem for forced Lagrangian systems)

A vector field X on Q is a symmetry of the forced Lagrangian (L, α) if and only if $X^v(L)$ is a constant of the motion.

Example (Fluid resistance)

- Consider a body of mass m moving along 1 dimension through a fluid that fully encloses it.
- The Rayleigh potential associated to the drag force is

$$\mathcal{R} = \frac{k}{3}\dot{q}^3, \qquad k = \frac{1}{2} C A \rho; \qquad L = \frac{1}{2} m \dot{q}^2.$$

- Consider the vector field

$$X = e^{kq/m} \frac{\partial}{\partial q}.$$

- $X^c(L) = X^v(\mathcal{R}) \Longrightarrow X^v(L) = m e^{kq/m} \dot{q}$ is a constant of the motion.
- When $k = 0$, X is the generator of translations and the conservation of momentum is recovered.

Other symmetries

- A vector field $\tilde{X}$ on TQ is called a **dynamical symmetry** if

$$[\tilde{X}, \Gamma_{L,\alpha}] = 0.$$

- A vector field $\tilde{X}$ on TQ is called a **Cartan symmetry** if

$$\mathcal{L}_{\tilde{X}}\theta_L = \mathrm{d}f, \qquad \tilde{X}(E_L) + \alpha(\tilde{X}) = 0$$

- If $\mathcal{L}_{\tilde{X}}\theta_L = \mathrm{d}f$, then $\tilde{X} = A^i \frac{\partial}{\partial q^i} + B^i \frac{\partial}{\partial \dot{q}^i}$ is a Cartan symmetry if and only if $f - A^i \frac{\partial L}{\partial \dot{q}^i}$ is a constant of the motion.

Example (Harmonic oscillator with a gyroscopic force)

- Consider a 2-dim. harmonic oscillator subject to a gyroscopic force:

$$L = \frac{1}{2}\sum_{i=1}^{2}\left[\left(\dot{q}^i\right)^2 - \left(q^i\right)^2\right], \qquad \alpha = g\left(\dot{q}^2 \mathrm{d}q^1 - \dot{q}^1 \mathrm{d}q^2\right).$$

- Consider the vector field $\tilde{X}$ on TQ given by

$$\tilde{X} = \left(g\dot{q}^1 + q^2\right)\frac{\partial}{\partial q^1} + \left(g\dot{q}^2 - q^1\right)\frac{\partial}{\partial q^2} + \dot{q}^2\frac{\partial}{\partial \dot{q}^1} - \dot{q}^1\frac{\partial}{\partial \dot{q}^2}.$$

- $\tilde{X}$ is a Cartan symmetry of (L, α), and its constant of the motion is

$$\ell = q^1\dot{q}^2 - q^2\dot{q}^1 - \frac{g}{2}\left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2\right].$$

Momentum map

- Consider a Lie group action of G on Q and the lifted action on TQ.
- The **momentum map** $J\colon TQ \to \mathfrak{g}^*$ assigns a function on TQ for each $\xi \in \mathfrak{g}$:

$$J^\xi = \langle J, \xi \rangle = \frac{\partial L}{\partial \dot{q}^i} \xi^i \,,$$

  where $\xi_Q = \xi^i \frac{\partial}{\partial q^i}$ is the generator of the ξ-action on Q.
- The action of ξ on TQ is generated by

$$\xi_{TQ} = \xi_Q^c = \xi^i \frac{\partial}{\partial q^i} + \dot{q}^j \frac{\partial \xi^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i}$$

Example ($\mathrm{SO}(3)$ and angular momentum)

- The generators of rotations on $Q = \mathbb{R}^3$ are given by

$$\xi_{Q,i} = \varepsilon_{ijk} x^j \frac{\partial}{\partial x^k}, \qquad i, j, k = 1, 2, 3\,,$$

  e.g., $\xi_{Q,1} = y\partial_z - z\partial_y$.
- Then,

$$\xi_{TQ,i} = \varepsilon_{ijk} x^j \frac{\partial}{\partial x^k} + \varepsilon_{ijk} \dot{x}^j \frac{\partial}{\partial \dot{x}^k}\,.$$

- Thus,

$$J^{\xi_i} = \varepsilon_{ijk} x^j \frac{\partial L}{\partial \dot{x}^k} =: L_k\,.$$

Group actions and quotient manifolds

- **Idea**: reducing the dimensions of TQ (i.e., taking out redundant d.o.f.) when L and α are G-invariant.
- Let
$$[x] = \{y \in M \mid \Phi(g, x) = y \text{ for some } g \in G\} ,$$
$$M/G = \{[x] \mid x \in M\} .$$
- Φ smooth, free and proper $\implies$ M/G is a differentiable manifold of dimension $\dim M - \dim G$.

Level sets

- Suppose that L and α are both left invariant by the $\mathfrak{g}$-action.
- Then, J^ξ is a constant of the motion $\forall \xi \in \mathfrak{g}$.
- Therefore the integral curves of $\Gamma_{L,\alpha}$ are contained in level sets $J^{-1}(\mu) \subset TQ$.
- In other words, a trajectory starting on a particular level set $J^{-1}(\mu)$ will not go out of the level set along the motion.

Theorem

- *Consider a* $\mathfrak{g}$*-invariant forced Lagrangian system* (L, α) *on* TQ*.*
- *Let* μ *be a regular value of* J *which is left invariant by the coadjoint action of the subgroup* $G_\mu \subseteq G$*.*
- *Then,* (L, α) *defines a reduced forced Lagrangian system* (L_μ, α_μ) *on* $(TQ)_\mu := J_\alpha^{-1}(\mu)/G_\mu$*.*
- *The reduced Lagrangian is related with the original one via*

$$L_\mu\left([q, \dot{q}]\right) = L\left(q, \dot{q}\right) \,,$$

and similarly for the reduced external force.

References for Symmetries and Reduction

- M. de León, M. Lainz, and A. L.-G., "Symmetries, constants of the motion, and reduction of mechanical systems with external forces," **J. Math. Phys.**, vol. 62, no. 4, p. 042 901, Apr. 2021.
- M. de León and P. R. Rodrigues, **Methods of Differential Geometry in Analytical Mechanics**, ser. North-Holland Mathematics Studies 158. North-Holland, 1989.
- C. Godbillon, **Géométrie Différentielle et Mécanique Analytique**, ser. Collection Méthodes. Hermann, 1969.
- A. L.-G., "The geometry of Rayleigh dissipation," Master's thesis, Universidad Autónoma de Madrid, Jul. 2021. arXiv: 2107.03780.
- J.-P. Ortega and T. S. Ratiu, **Momentum Maps and Hamiltonian Reduction**. Boston, MA: Birkhäuser Boston, 2004.

Forced Hamilton equations

- A **forced Hamiltonian system** is a pair (H, β), where $\beta = \beta_i(q,p)\mathrm{d}q^i$ is a semibasic 1-form on T^*Q.
- The forced dynamical vector field $X_{H,\beta}$ is given by

$$X_{H,\beta} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + \beta_i\right)\frac{\partial}{\partial p_i}$$

- Its integral curves satisfy the **forced Hamilton equations**:

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i},$$
$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i} - \beta_i.$$

Standard Hamilton-Jacobi problem

- The Hamilton-Jacobi problem consists in finding a **generating function** S on Q such that

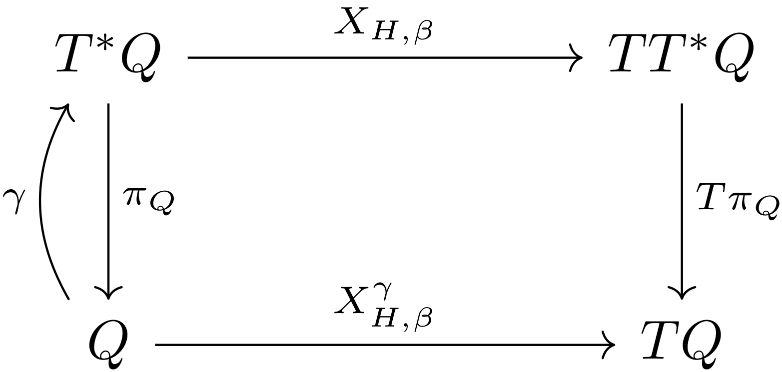
$$H\left(q^i, \frac{\partial S}{\partial q^i}\right) = E.$$

- Geometrically, this equation can be written as

$$\mathrm{d}\left(H \circ \gamma\right) = 0,$$

with $\gamma = \mathrm{d}S$ a 1-form on Q.

Geometric Hamilton-Jacobi problem



Hamilton-Jacobi problem for (H, β)

Theorem

Let γ be a closed 1-form on Q. Then the following conditions are equivalent:

1. $\mathrm{d}(H \circ \gamma) = -\gamma^* \beta$,
2. $\frac{\partial H}{\partial q^i} + \frac{\partial H}{\partial p_j} \frac{\partial \gamma_j}{\partial q^i} + \beta_i \circ \gamma = 0, \qquad i = 1, \ldots, n,$
3. *if $\sigma : \mathbb{R} \to Q$ is an integral curve of $X_{H,\beta}^{\gamma}$, then $\gamma \circ \sigma$ is an integral curve of $X_{H,\beta}$;*

If γ satisfies these conditions, it is called a solution of the Hamilton-Jacobi problem for (H, β).

Complete solutions I

- A map $\Phi : Q \times \mathbb{R}^n \to T^*Q$ is called **complete solution of the Hamilton-Jacobi problem** for (H, β) if
  1. Φ is a local diffeomorphism,
  2. for any $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, the map

$$\Phi_\lambda : Q \to T^*Q$$
$$q \mapsto \Phi_\lambda(q) = \Phi(q, \lambda_1, \ldots, \lambda_n)$$

  is a solution of the Hamilton-Jacobi problem for (H, β).

Complete solutions II

- Consider the functions given by

$$f_a = \pi_a \circ \Phi^{-1} : T^*Q \to \mathbb{R},$$

  where π_a denotes the projection over the a-th component of $\mathbb{R}^n$.
- The functions f_a are constants of the motion. Moreover, they are in involution, i.e.,

$$\{f_a, f_b\} = 0$$

Example

- Consider a n-dimensional forced Hamiltonian system (H, β), with

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2, \qquad \beta = \sum_{i=1}^{n} \kappa_i p_i^2 \mathrm{d}q_i.$$

- A complete solution of the Hamilton-Jacobi problem is

$$\Phi_\lambda = \sum_{i=1}^{n} \lambda_i e^{-\kappa_i q^i} \mathrm{d}q^i,$$

  with local generating function $S_\lambda = -\sum_{i=1}^{n} \frac{\lambda_i}{\kappa_i} e^{-\kappa_i q^i}$.

- The functions

$$f_a = e^{\kappa_a q^a} p_a, \qquad a = 1, \dots, n.$$

  are constants of the motion in involution.

Reduction and reconstruction of the Hamilton-Jacobi problem

- Let (H, β) be a forced Hamiltonian system on T^*Q.
- Let G be a Lie group that acts freely and properly on Q, and on T^*Q by the cotangent lift action.
- Suppose that this action preserves H and β.
- Then, we can introduce a reduced Hamiltonian $\tilde{H}$ and a reduced external force $\tilde{\beta}$ on $T^*(Q/G)$.
- If γ is a G-invariant solution of the Hamilton-Jacobi problem for (H, β), then it induces a solution $\tilde{\gamma}$ of the Hamilton-Jacobi problem for $(\tilde{H}, \tilde{\beta})$.
- Conversely, we can reconstruct γ from $\tilde{\gamma}$.

Example (Calogero-Moser system with a linear Rayleigh force)

- Consider a forced Hamiltonian system (H, β) on $T^*\mathbb{R}^2$, where

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + \frac{1}{(x-y)^2}\right), \qquad \beta = (p_x + p_y)(\mathrm{d}x - \mathrm{d}y).$$

- Consider the action $\Phi\left(t, (x, y)\right) = (t + x, t + y)$ of $\mathbb{R}$ on $\mathbb{R}^2$.
- Clearly, (H, β) is invariant under Φ^{T^*}. The momentum map is $J(x, y, p_x, p_y) = p_x + p_y$.
- We can identify $J^{-1}(\mu)/\mathbb{R}$ with $\mathbb{R}^2$, with coordinates (q, p) and the natural projection $\pi : (x, y, p, \mu - p) \mapsto (x - y, p)$.
- $\tilde{\gamma}_\lambda = \mathrm{d}\tilde{S}_\lambda \rightsquigarrow \gamma_\lambda = \mathrm{d}S_\lambda$, where the generating functions are

$$\tilde{S}_\lambda(q) = \frac{1}{2}q^2 - \frac{1}{2\mu q} + \lambda q, \qquad S_\lambda(x, y) = \tilde{S}_\lambda(x - y) + \mu y.$$

References for Hamilton-Jacobi Theory

- V. I. Arnold, **Mathematical Methods of Classical Mechanics**, ser. Graduate Texts in Mathematics. New York: Springer-Verlag, 1978.
- J. F. Cariñena, X. Gràcia, G. Marmo, E. Martínez, M. C. Muñoz-Lecanda, and N. Román-Roy, "Geometric Hamilton-Jacobi Theory," **Int. J. Geom. Methods Mod. Phys.**, vol. 03, no. 07, pp. 1417–1458, Nov. 2006.
- M. de León, M. Lainz, and A. L.-G., "Geometric Hamilton-Jacobi theory for systems with external forces," **J. Math. Phys.**, vol. 63, no. 2, p. 022 901, Feb. 2022.
- D. Iglesias-Ponte, M. de León, and D. M. de Diego, "Towards a Hamilton-Jacobi theory for nonholonomic mechanical systems," **J. Phys. A: Math. Theor.**, vol. 41, no. 1, p. 015 205, Jan. 2008.

Merci!

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