The inverse problem for contact Hamiltonian vector fields and applications to stability

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Outline of the talk

1. Review of contact geometry
2. The inverse problem
3. Stability
4. Stability

Review of contact geometry

Maximally non-integrable distributions

Definition

A distribution $D \subset \mathrm{T}M$ on a manifold M is **maximally non-integrable** if the bilinear map

$$\nu_D \colon D \times_M D \ni (X, Y) \mapsto \gamma\Big([X, Y]\Big) \in \mathrm{T}M/D$$

is non-degenerate. Here $[\cdot, \cdot]$ denotes the Lie bracket of vector fields with image in D, and $\gamma \colon \mathrm{T}M \to \mathrm{T}M/D$ is the canonical projection.

Contact distributions

Definition

Let M be a $(2n+1)$-dimensional manifold. A **contact distribution** C on M is a maximally non-integrable distribution of corank 1. The pair (M, C) is called a **contact manifold**.

Distributions as kernels of 1-forms

- Note that a distribution D of corank 1 on M can be locally written as the kernel of a (local) 1-form α on M.
- It is easy to see that D is integrable iff

$$\alpha \wedge \mathrm{d}\alpha = 0$$

  for any local 1-form α such that $D = \ker \alpha$.
- On the contrary, D is maximally non-integrable iff

$$\alpha \wedge \mathrm{d}\alpha^n = \alpha \wedge \underbrace{\mathrm{d}\alpha \wedge \cdots \wedge \mathrm{d}\alpha}_{n \text{ times}} \neq 0$$

  for any local 1-form α such that $D = \ker \alpha$.

Proposition

Let α be a one-form on a manifold M. The following statements are equivalent:

1. $\ker \alpha$ *is a contact distribution on* M,
2. $\alpha \wedge \mathrm{d}\alpha^n$ *is a volume form on* M,
3. $\flat_\alpha \colon \mathrm{T}M \ni v \mapsto \alpha(v)v + \iota_v \mathrm{d}\alpha \in \mathrm{T}^*M$ *is a VB-isomorphism,*
4. $\mathrm{T}M = \ker \alpha \oplus \ker \mathrm{d}\alpha$,
5. $\omega = \mathrm{d}(r\alpha)$ *is a symplectic form on* $M \times \mathbb{R} \setminus \{0\}$.

Moreover, $\dim M = 2n + 1$.
Additionally, note that these conditions are satisfied by α iff they are satisfied by $f\alpha$ for any nowhere-vanishing $f \in \mathscr{C}^\infty(M)$.

Definition

A one-form α satisfying the conditions above is called a **contact form**, and (M, α) is called a **co-oriented contact manifold**.

Remark (Not existence and not uniqueness of contact forms)

- Not all contact manifolds are co-orientable. Nevertheless, there always exists a co-orientable double covering space.
- In this talk, I will only be interested in local problems, so we can simply work with contact forms in $\mathbb{R}^{2n+1}$.
- A co-orientable contact distribution C does not fix the contact form α, but rather the equivalence class

$$\alpha \sim \tilde{\alpha} \iff \ker \alpha = \ker \tilde{\alpha} \iff \exists\, f \colon M \to \mathbb{R} \setminus \{0\} \text{ such that } \tilde{\alpha} = f\alpha\,.$$

Definition

Let α be contact form on M. The **Reeb vector field** R is defined by $\flat_\alpha(R) = \alpha$. Equivalently, R is the unique section of $\ker \mathrm{d}\alpha$ such that $\alpha(R) = 1$.

Note that we can decompose $\mathsf{T}M$ into the contact distribution and the line spanned by R:

$$\mathsf{T}M = \ker \alpha \oplus \operatorname{span} R\,.$$

Proposition

Let α be contact form, and X a vector field on a manifold M. The following assertions are equivalent:

1. *The flow φ_t of X preserves the contact distribution $C = \ker \alpha$, i.e. $\mathrm{T}\varphi_t C \subseteq C$,*
2. $[X, Y] \in \Gamma(C)$ *for all* $Y \in \Gamma(C)$,
3. $\exists f \in \mathscr{C}^\infty(M)$ *such that* $\mathcal{L}_X \alpha = f \alpha$,
4. $\mathcal{L}_X \alpha = -R(h)\, \alpha$, *where* $h := -\alpha(X)$,
5. $\flat_\alpha(X) = \mathrm{d}h - (Rh + h)\, \alpha$, *where* $h := -\alpha(X)$.

Moreover, conditions ③,④ *and* ⑤ *hold for* α *iff they hold for all* $g\alpha$ *for any nowhere-vanishing* $g \in \mathscr{C}^\infty(M)$.

Definition

If X satisfies the conditions above, it is called a **contact Hamiltonian vector field**. If the contact form α is fixed, we denote $X_h := X$, where $h = -\alpha(X)$.

Definition

A **contact Hamiltonian system** is a triple (M, α, h), where α is a contact form on M and $h \in \mathscr{C}^\infty(M)$ a fixed **Hamiltonian function**.

- The dynamics of (M, α, h) is determined by the flow of X_h.
- Note that $\left(M, f\alpha, \frac{h}{f}\right)$ determine the same geometry $(\ker \alpha)$ and dynamics (X_h) for any nowhere-vanishing $f \in \mathscr{C}^\infty(M)$.

The inverse problem

Problem (The (local) inverse problem for contact Hamiltonian vector fields)

Let X be a vector field on a manifold M^{2n+1}, and $x \in M$. Does it exist some (local) contact distribution $C \subset \mathsf{T}U$ on a neighbourhood $U \subseteq M$ of x making X a contact Hamiltonian vector field?

Non-vanishing vector field

- If $X(x) \neq 0$, the answer is affirmative.
- Indeed, by the straightening theorem, we can construct a chart $(U; x^a)$ around x such that $X = \partial_{x^0}$.
- We can relabel these coordinates as

$$z = x^0 \,, \quad q^i = x^i \,, \quad p_i = x^{i+n} \,, \quad 1 \leq i \leq n \,,$$

so that they are Darboux coordinates for the contact form $\alpha = \mathrm{d}z - p_i \mathrm{d}q^i$ and $X = R = \partial_z$ is the corresponding Reeb vector field.

Equilibrium point

- If $X(x) = 0$, i.e. x is an equilibrium point of X, the problem is quite involved, and its answer may be positive or negative.
- Let us for now focus on the case of **linear** vector fields, that is, around x there is a system of coordinates (x^i) in which X reads

$$X = \sum_{i,j=0}^{2n} A^i_j \, x^j \, \partial_{x^i} \,,$$

  where $\mathrm{A} = D_x X = (A^i_j)$ is a real matrix.
- The properties of this matrix determine whether X can be a contact Hamiltonian vector field or not.

Theorem (de Lucas, L. G.)

Let X be a linear vector field on $\mathbb{R}^{2n+1}$, and let $A = DX|_0$. Then, X is a contact vector field on a neighbourhood of 0 (with respect to some contact form) if and only if the following conditions are satisfied:

1. *The tangent space $\mathsf{T}_0\mathbb{R}^{2n+1}$ can be decomposed into an A-invariant line $\ell \subset \mathsf{T}_0\mathbb{R}^{2n+1}$ and an a complementary A-invariant subspace $\xi_0 \subset \mathsf{T}_0\mathbb{R}^{2n+1}$:*

$$\mathsf{T}_0\mathbb{R}^{2n+1} = \xi_0 \oplus \ell, \qquad A(\xi_0) \subset \xi_0, \quad A(\ell) \subset \ell\,.$$

2. *The spectrum of A satisfies the resonance relation*

$$\lambda_i + \lambda_j = \lambda_\ell\,, \quad \forall\, i \neq j\,,$$

where λ_i and λ_j denote eigenvalues of $A|_{\xi_0}$ counted with multiplicities, and λ_ℓ denotes the eigenvalue of $A|_\ell$.

These conditions can be equivalently expressed as follows:

1. At least one block from the Jordan normal form of the matrix A size 1×1:

$$A = \left(\begin{array}{c|c|c} \ddots & & \\ \hline & \ddots & \\ \hline & & \underbrace{A|_{\ell}}_{1\times 1} \end{array}\right) = \left(\begin{array}{c|c} \overbrace{A|_{\xi_0}}^{2n\times 2n} & \\ \hline & \underbrace{A|_{\ell}}_{1\times 1} \end{array}\right)$$

2. The spectrum of A satisfies the resonance relation

$$\lambda_i + \lambda_j = \lambda_\ell\,, \quad \forall\, i \neq j\,,$$

where λ_i and λ_j denote eigenvalues of $A|_{\xi_0}$ counted with multiplicities, and λ_ℓ denotes the eigenvalue of $A|_\ell$.

Lemma

Let X be a vector field on a manifold M with an equilibrium point at $x \in M$. Then, for any k-form β on M and any vectors $v_1, \ldots, v_k \in \mathrm{T}_x M$,

$$\left(\mathcal{L}_X \beta\right)_x \left(v_1, \ldots, v_k\right) = \beta_x \left(A v_1, \ldots, v_k\right) + \cdots + \beta_x \left(v_1, v_2 \ldots, A v_k\right) \ ,$$

where $A \coloneqq DX|_x$.
In particular,

$$\left(\mathcal{L}_X \alpha\right)_x = \alpha_x \circ A$$

for a 1-form α.

Proposition

Let X be a linear vector field on $\mathbb{R}^{2n+1}$ such that $A = DX|_0$ is a nilpotent matrix, that is, there exist a system of linear coordinates (x^i) around 0 in which the vector field X reads

$$X = \sum_{i=1}^{k} x^{i+1}\, \partial_{x^i}\,, \quad 0 \leq k \leq 2n\,.$$

Then, there exists no contact form around 0 with respect to which X is a contact Hamiltonian vector field (except the trivial case $X \equiv 0$).

Proof.

All the solutions α around 0 of the PDE $\mathcal{L}_X \alpha = f\alpha$ for an arbitrary $f \in \mathscr{C}^\infty(M)$ have

$$\ker \alpha \cap \ker \mathrm{d}\alpha \neq \{0\}\,.$$

□

Proof of the theorem: conditions are necessary

- Let α be a contact form near $0 \in \mathbb{R}^{2n+1}$ such that $\mathcal{L}_X \alpha = f\alpha$ for some function f defined around 0.
- By the previous lemma,

$$\mathrm{d}\alpha_0(Av, w) + \mathrm{d}\alpha_0(v, Aw) = f(0)\, \mathrm{d}\alpha_0(v, w), \qquad \forall v, w \in \xi_0\,.$$

- Since $\mathrm{d}\alpha_0|_{\xi_0}$ is symplectic, if v and w are two linearly independent eigenvectors of $A|_{\xi_0}$ with eigenvalues λ_i and λ_j, we have

$$\lambda_i + \lambda_j = f(0)\,.$$

- On the other hand

$$f(0) = \iota_{R_0} f(0)\alpha_0 = \iota_{R_0} \left(\mathcal{L}_X \alpha\right)_0 = \alpha_x \circ A(R_0)$$

Proof of the theorem: conditions are necessary

- Since X is a contact Hamiltonian vector field, $A(\xi_0) \subseteq \xi_0$.
- Because the $2n$-dimensional subspace ξ_0 is A-invariant, the only possibilities for the Jordan normal form of A are as follows:
  - A has (at least) a 1×1 Jordan block:
    $$A = \left(\begin{array}{c|c} A|_{\xi_0} & 0 \\ \hline 0 & A|_{\ell} \end{array}\right).$$
    As $\xi_0 = \ker \alpha_0$, we have that
    $$f(0) = \alpha_x \circ A(R_0) = \alpha_x \circ A|_{\ell}(R_0) = \lambda_\ell \,.$$
  - A is nilpotent $\rightsquigarrow$ excluded by the previous proposition

Proof of the theorem: conditions are sufficient

- Choose linear coordinates (x^i, z) around 0 such that $\xi_0 = \mathrm{span}\{\partial_{x^i}\}$ and $\ell = \mathrm{span}\{\partial_z\}$. Hence,

$$A = \begin{pmatrix} B & 0 \\ 0 & \lambda_\ell \end{pmatrix}, \quad X = \sum_{i,j=1}^{2n} B_{ij} x^i \partial_{x^j} + \lambda_\ell z \, \partial_z \,.$$

- Consider the one-forms

$$\alpha = \mathrm{d}z + \theta \,, \quad \theta = \sum_{i,j=1}^{2n} C_{ij} x^i \mathrm{d}x^j \,, \quad C_{ij} = \mathrm{const.}$$

- Note that α is a contact form iff the matrix (C_{ij}) is non-singular.

Proof of the theorem: conditions are sufficient

- From routine computations, we can see that $\mathcal{L}_X \alpha = \lambda_\ell \, \alpha$ iff

$$C B^t C^{-1} = \lambda_\ell \mathbb{1} - B \,,$$

- This means that B is similar to $\lambda_\ell \mathbb{1} - B$, i.e. B and $\lambda_\ell \mathbb{1} - B$ have the same spectrum with the same geometric multiplicities, which is precisely the resonance condition.

If X is a vector field with $X(0) = 0$ and $A = DX|_0$, its linearization is

$$X_{\mathrm{lin}} = \sum_{i,j} A^i_j x^i \partial_{x^j}$$

Corollary (Linearization preserves the contact property)

Let ξ be a (local) contact distribution on $\mathbb{R}^{2n+1}$, and let X be a (local) contact Hamiltonian vector field with an equilibrium point at 0. Then the linearization X_{lin} of X is a contact Hamiltonian vector field near 0 with respect to some local contact distribution ξ_{lin}, such that $\xi_{\mathrm{lin}}|_0 = \xi_0$.

Corollary

Let X be a linear vector field on $\mathbb{R}^3$ with an equilibrium point at 0. Then, there exists a system of linear coordinates (x, y, z) centered in 0 in which X has one of the expressions in the left-hand side of the table. Moreover, X is a contact Hamiltonian vector field with respect to the (local) contact form $\alpha = \mathrm{d}z + y\mathrm{d}x - x\mathrm{d}y$ iff the corresponding condition in the right-hand side is satisfied:

Vector field X	*Contact iff*
$k_1 x\partial_x + k_2 y\partial_y + k_3 z\partial_z$	$k_3 = k_1 + k_2$ *or a permutation*
$(k_1 x + y)\partial_x + k_1 y\partial_y + k_2 z\partial_z$	$k_2 = 2k_1$ *or* $k_1 = 2k_2$
$(k_1 x + y)\partial_x + (k_1 y + z)\partial_y + k_1 z\partial_z$	*never*
$-y\partial_x + x\partial_y + k_3 z\partial_z$	$k_3 = 0$

Example: Lorenz system

- The Lorenz system is given by the following system of ordinary differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z\,.$$

- In other words, the trajectories of the system are given by the flow of the vector field

$$X = \sigma(y - x)\partial_x + \big(x(\rho - z) - y\big)\partial_y + (xy - \beta z)\partial_z\,.$$

- It has an equilibrium point at 0.
- The linerisation of X around 0 is given by

$$X_{\mathrm{lin}} = \sigma(y - x)\partial_x + (\rho x - y)\partial_y - \beta z\partial_z\,.$$

Example: Lorenz system

- Then,

$$A = DX\big|_0 = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix},$$

whose eigenvalues are

$$\lambda_1 = \frac{1}{2}\left(-\sqrt{4\rho\sigma + \sigma^2 - 2\sigma + 1} - \sigma - 1\right),$$
$$\lambda_2 = \frac{1}{2}\left(\sqrt{4\rho\sigma + \sigma^2 - 2\sigma + 1} - \sigma - 1\right), \quad \lambda_3 = -\beta\,.$$

- Since A is diagonalisable, X_{lin} is a contact vector field with respect to some local contact form iff

$$\lambda_i + \lambda_j = \lambda_k\,,$$

where (i, j, k) is some permutation of $(1, 2, 3)$.

Example: Lorenz system

- For instance, if $\sigma = \beta - 1$, then $\lambda_1 + \lambda_2 = \lambda_3$. In that case, the change of basis of $\mathrm{T}_0\mathbb{R}^3$ diagonalising A leads to the following linear change of coordinates

$$\xi = \frac{y\left(\sqrt{\sigma(4\rho+\sigma-2)+1} - \sigma + 1\right) - 2\rho x}{2\sqrt{\sigma(4\rho+\sigma-2)+1}}\,,$$
$$\chi = \frac{2\rho x + y\left(\sqrt{\sigma(4\rho+\sigma-2)+1} + \sigma - 1\right)}{2\sqrt{\sigma(4\rho+\sigma-2)+1}}\,, \quad \zeta = z\,,$$

such that

$$X_{\mathrm{lin}} = \lambda_1 \xi \partial_\xi + \lambda_2 \chi \partial_\chi + (\lambda_1 + \lambda_2)\zeta\partial_\zeta\,.$$

Example: Lorenz system

- By the corollary of the previous slide, X_{lin} is a contact Hamiltonian vector field with respect to $\alpha_{\mathrm{lin}} = \mathrm{d}\zeta + \chi \mathrm{d}\xi - \xi \mathrm{d}\chi$. The corresponding Hamiltonian function is

$$h_{\mathrm{lin}} = -\alpha_{\mathrm{lin}}(X_{\mathrm{lin}}) = (\lambda_2 - \lambda_1)\xi\chi - (\lambda_1 + \lambda_2)\zeta \,.$$

Stability

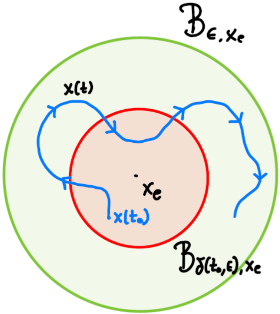
Equilibrium points

- Let M be an n-dimensional manifold
- The solutions of the system of ODEs

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = X^i(x)\,, \qquad i = 1, \ldots, n\,,$$

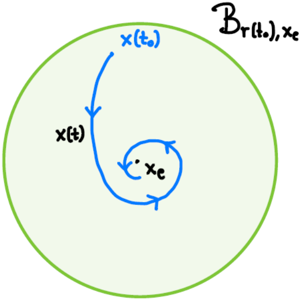
  are the integral curves of the vector field $X = X^i \frac{\partial}{\partial x^i}$.
- An **equilibrium point** is a point $x_e \in M$ such that $X(x_e) = 0$.

Stable equilibrium points



If $M = \mathbb{R}^n$, an equilibrium point x_e of X is called **stable** if, for every $t_0 \in \mathbb{R}$ and any ball B_{ϵ,x_e}, there exists a ball $B_{\delta(\epsilon),x_e}$, such that every integral curve $x(t)$ of X with $x(t_0) \in B_{\delta(\epsilon),x_e}$ satisfies that $x(t) \in B_{\epsilon,x_e}$ for all times $t \geq t_0$.

Asymptotically stable equilibrium points



An equilibrium point $x_e \in \mathbb{R}^n$ is **asymptotically stable** if x_e is stable and there exists an open neighbourhood B_{r,x_e} of x_e such that every integral curve $x(t)$ of X with some t_0 satisfying $x(t_0) \in B_{r,x_e}$ converges to x_e.

How to extend this to manifolds?

- The existence of partitions of unity implies that every differentiable manifold can be endowed with a Riemannian metric induced by the Euclidean metric.
- Moreover, the topology induced by the Riemannian metric coincides with the topology of the manifold.
- This implies that a coordinate neighbourhood U is homeomorphic to an open subset in $\mathbb{R}^n$ with the Euclidean norm.
- We will identify balls in $\mathbb{R}^n$ with the neighbourhoods in U to which they are homeomorphic.

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Lyapunov functions

Theorem

Let $X \in \mathfrak{X}(M)$ be a vector field such that $X(x_0) = 0$. If there exists a function $V : U \to \mathbb{R}$, defined on some open neighbourhood U of x_0 such that

1. $V(x_0) = 0$ *and* $V(x) > 0$ *for* $x \in U \setminus \{x_0\}$,
2. $\dot{V}(x) = (XV)(x) \leq 0$ *for* $x \in U \setminus \{x_0\}$,

then x_0 is stable. If additionally $\dot{V}(x) < 0$ for $x \in U \setminus \{x_0\}$, then x_0 is asymptotically stable.

Lyapunov functions

Definition

A function $V\colon U \to \mathbb{R}$ satisfying ① and ② is called a **Lyapunov function**. If $\dot{V}(x) < 0$ for $x \in U \setminus \{x_0\}$, the function V is called a **strict Lyapunov function**.

Jacobi structure associated with a contact form

- A contact form α on M defines a Jacobi bracket

$$\begin{aligned} &\{\cdot,\cdot\}_\alpha \colon \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M) \\ &\{f, g\}_\alpha = X_f(g) + gR(f) = -\alpha\big([X_f, X_g]\big)\,. \end{aligned}$$

- The map $f \mapsto X_f$, with inverse $X \mapsto -\alpha(X)$, is a Lie algebra anti-isomorphism between $(\mathscr{C}^\infty(M), \{\cdot,\cdot\}_\alpha)$ and the algebra of contact Hamiltonian vector fields with $[\cdot,\cdot]$.
- If α and $\tilde{\alpha}$ generate the same contact distribution, their associated Jacobi brackets are conformal (in the sense of Dazord, Lichnerowicz, and Marle).

Dissipated quantities of a contact Hamiltonian system

Proposition

Let C be a contact distribution on M, and let X_1, X_2 be contact Hamiltonian vector fields. For any contact form α (locally) generating C, let R_α denote its Reeb vector field, and let $f_i^\alpha := -\alpha(X_i)$. The following statements are equivalent:

1. $[X_1, X_2] \in \Gamma(C)$,
2. $\{f_1^\alpha, f_2^\alpha\}_\alpha = 0$,
3. $\mathcal{L}_{X_1} f_2 = -R_\alpha(f_1^\alpha) f_2$,

Definition

Let (M, α, h) be a contact Hamiltonian system. A dissipated quantity is a function $f \in \mathscr{C}^\infty(M)$ satisfying $\{f, h\}_\alpha = 0$.

Dissipated quantities and stability

Proposition (de Lucas, L. G.)

Let (M, α, h) *be a contact Hamiltonian system such that* $X_h(x_0) = 0$*. Suppose that* $f_1, \ldots, f_k$ *are dissipated quantities. If* $(Rh)(x_0) > 0$ *at an isolated point* $x_0 \in \bigcap_{i=1}^k f_i^{-1}(0)$*, then* x_0 *is asymptotically stable.*

Proof.

There exists a neighbourhood U of x_0 where $Rh > 0$ and such that $\bigcap_{i=1}^k f_i^{-1}(0) \cap U = \{x_0\}$. By construction,

$$V : x \in U \mapsto \sum_{i=1}^{k} f_i^2(x) \in \mathbb{R}$$

is a strict Lyapunov function. □

Dissipated quantities and stability

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Necessary condition for being an isolated point

Proposition

Let $f_1, \ldots, f_k \in \mathscr{C}^\infty(M)$ *be such that* $f_i(x_0) = 0$ *for* $i = 1, \ldots, k$ *and* $\dim M \geq k + 1$*. If* x_0 *is an isolated point of* $\bigcap_{i=1}^k f_i^{-1}(0)$*, then*

$$\mathrm{d}f_1\big|_{x_0} \wedge \cdots \wedge \mathrm{d}f_k\big|_{x_0} = 0.$$

Necessary condition for being an isolated point

Proof.

Suppose that $\mathrm{d}f_1|_{x_0} \wedge \cdots \wedge \mathrm{d}f_k|_{x_0} \neq 0$. Then, on some neighbourhood U of x_0, the map $\Phi\colon U \ni x \mapsto (f_1(x), \ldots, f_k(x)) \in \mathbb{R}^k$ is regular, and hence $\mathrm{d}f_1|_U \wedge \cdots \wedge \mathrm{d}f_k|_U \neq 0$. Thus,

$$\Phi^{-1}(0) = f_1^{-1}(0) \cap f_2^{-1}(0) \cap \cdots \cap f_k^{-1}(0) \cap U$$

is a k-codimensional submanifold and x_0 is not an isolated point of $\bigcap_{i=1}^k f_i^{-1}(0)$. □

Sufficient condition for being an isolated point

Proposition

Let $f_1, \dots, f_k \in \mathscr{C}^\infty(M)$ *with* $k < n$ *be such that* $f_i(x_0) = 0 \; \forall\, i = 1, \dots, k$, *and* $\dim \langle \mathrm{d} f_i|_{x_0} \rangle = k-1$. *W.l.o.g., assume that* $\mathrm{d}f_1|_{x_0}, \dots, \mathrm{d}f_{k-1}|_{x_0}$ *are linearly independent. If* $g = f_k + \lambda_1 f_1 + \cdots + \lambda_{k-1} f_{k-1}$, *where* $\lambda_1, \dots, \lambda_{k-1}$ *are Lagrange multipliers, has a strict minimum or maximum at* x_0, *then* x_0 *is an isolated point of* $\bigcap_{i=1}^k f_i^{-1}(0)$.

Sufficient condition for being an isolated point

Proof.

By construction, $g(x_0) = 0$. If x_0 is a constrained local strict minimum or maximum of g, then there exists a neighbourhood U of x_0 in $f_1^{-1}(0) \cap \ldots \cap f_{k-1}^{-1}(0)$ such that $g(x) \neq 0$ for all $x \in U \setminus \{x_0\}$. Consequently, $f_1(x), \ldots, f_k(x)$ cannot vanish simultaneously at any $x \in U \setminus \{x_0\}$. We conclude that

$$\bigcap_{i=1}^{k} f_i^{-1}(0) \cap \widehat{U} = \{x_0\}$$

for any open subset $\widehat{U}$ in M such that $\widehat{U} \cap \bigcap_{i=1}^{k-1} f_i^{-1}(0) = U$. □

Theorem (de Lucas, L. G.)

Let (M, α, h) be a contact Hamiltonian system and let x_0 be an equilibrium point of X_h. Suppose that $f_1, \dots, f_k \in \mathscr{C}^\infty(M)$ are dissipated quantities for X_h such that $f_i(x_0) = 0$ for $i = 1, \dots, k$, and $\dim\langle \mathrm{d}f_1|_{x_0}, \dots, \mathrm{d}f_k|_{x_0}\rangle = k - 1$. W.l.o.g., assume that $\mathrm{d}f_1|_{x_0}, \dots, \mathrm{d}f_{k-1}|_{x_0}$ are linearly independent. If the function $g = f_k + \lambda_1 f_1 + \cdots + \lambda_{k-1} f_{k-1}$, where $\lambda_1, \dots, \lambda_{k-1}$ are Lagrange multipliers, has a strict minimum or maximum at x_0, then x_0 is asymptotically stable.

Computation of the Lagrange multipliers

Since $\mathrm{d}f_1|_{x_0}, \ldots, \mathrm{d}f_{k-1}|_{x_0}$ are indep., and $\dim\langle \mathrm{d}f_1|_{x_0}, \ldots, \mathrm{d}f_k|_{x_0}\rangle = k-1$, one has that

$$\mathrm{d}f_k|_{x_0} = -\sum_{i=1}^{k-1} \lambda_i \mathrm{d}f_i|_{x_0} .$$

Example

- Consider the contact Hamiltonian system $(\mathbb{R}^3, \alpha, h)$, with

$$\alpha = \mathrm{d}z - p\mathrm{d}q\,, \quad h = \frac{p^2}{2} + \frac{q^2}{2} + z\,.$$

- The Hamiltonian vector field of h is

$$X_h = p\frac{\partial}{\partial q} - (q+p)\frac{\partial}{\partial p} + \left(\frac{p^2}{2} - \frac{q^2}{2} - z\right)\frac{\partial}{\partial z}\,,$$

which vanishes at 0.
- The function $f = z - \frac{pq}{2}$ is a dissipated quantity.
- We have that $h^{-1}(0) \cap f^{-1}(0) = \{0\}$.
- Since $Rh = 1$ everywhere (in particular, $Rh(0) > 0$), it follows that 0 is an asymptotically stable equilibrium point of X_h.

Future research

- Inverse problem for non-linear vector fields around an equilibrium.
- Utilising dissipated quantities for studying the stability of more complicated examples.

Wielkie dzięki za uwagę!

Moltes gràcies por la atenció!

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🌐 These slides are available at www.alopezgordon.xyz