Reduction of hybrid Hamiltonian systems with non-equivariant momentum maps

Leonardo Colombo, María Emma Eyrea Irazú, María Eugenia García, Asier López-Gordón, and Marcela Zuccalli

7th International Conference on Geometric Science of Information

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES

Hybrid systems

- A hybrid dynamical system is one which combines continuous and discrete transitions.
- The dynamics of such systems are continuous “most of the time”, except at some instants at which abrupt changes occur.
- This framework may be used to model mechanical systems with impacts.

Hybrid systems

Definition

A **hybrid system** is a 4-tuple $\mathscr{H} = (D, X, S, \Delta)$, formed by

1. a manifold D,
2. a vector field $X \in \mathfrak{X}(D)$,
3. a submanifold $S \subset D$ of codimension 1 or greater,
4. an embedding $\Delta \colon S \to D$.

The dynamics generated by $\mathscr{H}$ are the curves $c \colon I \subseteq \mathbb{R} \to D$ such that

$$\begin{aligned} \dot{c}(t) &= X(c(t))\,, && \text{if } c(t) \notin S\,, \\ c^{+}(t) &= \Delta(c^{-}(t))\,, && \text{if } c(t) \in S\,, \end{aligned}$$

where

$$c^{\pm}(t) = \lim_{\tau \to t^{\pm}} c(\tau)\,.$$

Definition

A simple hybrid system $\mathcal{H} = (D, X, S, \Delta)$ is said to be a **simple hybrid Hamiltonian system** if $X = X_H$ is the Hamiltonian vector field associated with a Hamiltonian system (D, ω, H).

Definition

Let $\mathcal{H} = (D, X, S, \Delta)$ be a simple hybrid system. A Lie group action $\Phi : G \times D \to D$ of G on D, is called a **hybrid action** if

1. $\Phi|_{G\times S}$ is a Lie group action of G on S,
2. the impact map is equivariant with respect to this action, namely,

$$\Delta \circ \Phi_g\big|_S = \Phi_g \circ \Delta\,, \quad \forall g \in G\,.$$

Definition

Suppose that $\mathcal{H} = (D, X_H, S, \Delta)$ is a hybrid Hamiltonian system with associated Hamiltonian system (D, ω, H), and assume that the action Φ is hybrid and symplectic. A momentum map $J \colon D \to \mathfrak{g}^\star$ is called a **generalized hybrid momentum map** for $\mathcal{H}$ if, for each regular value μ_- of J, and each connected component C of S,

$$\Delta \left(J|_C{}^{-1}(\mu_-) \right) \subset J^{-1}(\mu_+),$$

for some regular value μ_+.

The tuple $(D, X_H, S, \Delta, \omega, \Phi, J)$ will be called a **hybrid Hamiltonian** G**-space**. We will call $\mu \in \mathfrak{g}^*$ a **hybrid regular value** if it is a regular value of both J and $J|_S$.

- Let $(D, X_H, S, \Delta, \omega, \Phi, J)$ be a hybrid Hamiltonian G-space.
- The **co-adjoint cocycle** associated with J is the map $\sigma : G \to \mathfrak{g}^*$ determined by

$$\langle \sigma(g), \xi \rangle = J_\xi(\Phi_g(x)) - \mathrm{Ad}^*_{g^{-1}}(J_\xi(x)) \,, \quad \forall\, \xi \in \mathfrak{g} \,, \quad \forall\, x \in D \,.$$

- The **affine action** of G on $\mathfrak{g}^*$ is given by

$$\Psi : (G, \mathfrak{g}^*) \ni (g, \mu) \mapsto \mathrm{Ad}^*_{g^{-1}} \mu + \sigma(g) \in \mathfrak{g}^* \,.$$

- By construction, the momentum map is equivalent w.r.t. this action, i.e.,

$$\Psi_g \circ J = J \circ \Phi_g\,, \quad \forall g \in G\,.$$

- Let $\tilde{G}_\mu$ denote the **isotropy subgroup** of $\mu \in \mathfrak{g}^*$ under the action Ψ, given by

$$\tilde{G}_\mu = \{g \in G : \Psi(g, \mu) = \mathrm{Ad}^*_{g^{-1}} \mu + \sigma(g) = \mu\}\,.$$

Proposition

Let $(D, X_H, S, \Delta, \omega, \Phi, J)$ *be a hybrid Hamiltonian G-space. Assume that G is connected. If* Δ *is equivariant with respect to* Φ*, and* μ_-*,* μ_+ *are regular values of* J *such that*

$$\Delta\left(J|_S{}^{-1}(\mu_-)\right) \subset J^{-1}(\mu_+)\,,$$

then the isotropy subgroups at μ_- *and at* μ_+ *under the action* Ψ *coincide, i.e.,* $\tilde{G}_{\mu_-} = \tilde{G}_{\mu_+}$.

Theorem

Let $(D, X_H, S, \Delta, \omega, \Phi, J)$ be a hybrid Hamiltonian G-space. Assume that G is connected, and consider a discrete sequence $\Lambda = \{\mu_i\}$ of regular values of J such that $\Delta\left(J|_S{}^{-1}(\mu_i)\right) \subset J^{-1}(\mu_{i+1})$. Let $\tilde{G}_{\mu_i} = \tilde{G}_{\mu_0}$ be the isotropy subgroup in μ_i (for any μ_i in the sequence) under the affine action. Assume that Φ and $\Phi|_{\tilde{G}_\mu \times J^{-1}(\mu)}$ are free and proper actions. Then, for any $\mu_i \in \Lambda$, we have the reduced hybrid system

$$(D_{\mu_i}, X_{H_{\mu_i}}, S_{\mu_i}, \Delta_{\mu_i}), \quad D_{\mu_i} := J^{-1}(\mu_i)/\tilde{G}_{\mu_i} .$$

The reduction scheme is summarized in the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & J^{-1}(\mu_i) \arrow[d] & J|_S{}^{-1}(\mu_i) \arrow[l, hook'] \arrow[d] \arrow[r, "\Delta|_{J^{-1}(\mu_i)}"] & J^{-1}(\mu_{i+1}) \arrow[d] & \cdots \arrow[l, hook'] \\
\cdots \arrow[r] & \frac{J^{-1}(\mu_i)}{\tilde{G}_{\mu_0}} & S_{\mu_i} \arrow[l, hook'] \arrow[r, "\Delta_{\mu_i}"] & \frac{J^{-1}(\mu_{i+1})}{\tilde{G}_{\mu_0}} & \cdots \arrow[l, hook']
\end{tikzcd}$$

A toy example

- Let $Q = \mathbb{R}^2$, and consider $\mathrm{T}^*Q \simeq \mathbb{R}^4$ endowed with the canonical symplectic form $\omega_Q = \mathrm{d}q^i \wedge \mathrm{d}p_i$, where (q^i, p_i) are bundle coordinates induced by the canonical coordinates (q^i) of Q.
- Consider the Lie group action $\Phi\colon \mathbb{R}^2 \times \mathrm{T}^*Q \to \mathrm{T}^*Q$ of $G = \mathbb{R}^2$ on T^*Q given by

$$\Phi_{(a,b)}\left(q^1, q^2, p_1, p_2\right) = \left(q^1 + a, q^2 + a, p_1 + b, p_2 + b\right) .$$

- The associated infinitesimal generators are

$$\xi_1^{\mathrm{T}^*Q} = \partial_{q^1} + \partial_{q^2} , \quad \xi_2^{\mathrm{T}^*Q} = \partial_{p_1} + \partial_{p_2} .$$

- Note that Φ is a symplectic action.

A toy example

- A momentum map $J\colon \mathrm{T}^*Q \to \mathfrak{g}^*$ for the action Φ is given by

$$J\left(q^1, q^2, p_1, p_2\right) = \left(p_1 + p_2, -q^1 - q^2\right) .$$

- Its co-adjoint cocycle is given by $\sigma(a, b) = (2b, -2a)$.
- The Hamiltonian function

$$H\left(q^1, q^2, p_1, p_2\right) = \frac{(p_1 - p_2)^2}{2} + V(q^1 - q^2)\,,$$

where V is a potential function depending only on $q^1 - q^2$, is Φ-invariant.

A toy example

- Consider the hybrid Hamiltonian system $\mathcal{H} = (D, X_H, S, \Delta)$, with X_H the Hamiltonian vector field of H, and

$$S = \left\{ \left(q^1, q^2, p_1, p_2\right) \mid q^1 - q^2 = c\,, \quad p_1 - p_2 < 0 \right\}\,,$$

$$\Delta\left(q^1, q^2, p_1, p_2\right) = \left(q^1, q^2, p_1 - \frac{1+e}{2}(p_1 - p_2), p_2 + \frac{1+e}{2}(p_1 - p_2)\right)\,,$$

  where $c \in \mathbb{R}$ and $e \in [0, 1]$.
- The action Φ is a hybrid action for $\mathcal{H}$, and J is a hybrid momentum map, i.e., $J \circ \Delta = J|_S$.
- The isotropy subgroup with respect to the affine action is trivial: $\tilde{G}_\mu = \{0\}$.

A toy example

- Let $\mu = (\mu_1, \mu_2) \in \mathfrak{g}^*$ be a regular value of J, and consider the quotient manifold $D_\mu = J^{-1}(\mu)/\tilde{G}_\mu = J^{-1}(\mu)$, where

$$\begin{aligned} J^{-1}(\mu) &= \left\{(q^1, q^2, p_1, p_2) : J(q^1, q^2, p_1, p_2) = \mu\right\}, \\ &= \left\{(q^1, q^2, p_1, p_2) : (p_1 + p_2, -(q^1 + q^2)) = (\mu_1, \mu_2)\right\}. \end{aligned}$$

- We can use $(q^2\big|_{D_\mu}, p_2|_{D_\mu})$ as coordinates in D_μ. With a slight abuse of notation, we will denote them simply by (q^2, p_2).

A toy example

- The reduced hybrid system is $\mathcal{H}_\mu = (D_\mu = J^{-1}(\mu), X_{H_\mu}, S_\mu, \Delta_\mu)$, where X_{H_μ} is the Hamiltonian vector field of

$$H_\mu(q^2, p_2) = \frac{(\mu_1 - 2p_2)^2}{2} + V(-\mu_2 - 2q^2)\,,$$

and

$$S_\mu = \left\{ \left(q^2, p_2, \mu_1, \mu_2\right) \mid -\mu_2 - 2q^2 = c\,, \quad \mu_1 - 2p_2 < 0 \right\}\,,$$

$$\Delta_\mu \left(q^2, p_2, \mu_1, \mu_2\right) = \Bigg(-\mu_2 - q^2, q^2, (\mu_1 - p_2) - \frac{1+e}{2}(\mu_1 - 2p_2), \\ p_2 + \frac{1+e}{2}(\mu_1 - 2p_2)\Bigg)\,.$$

Conclusions and outlook

- We have obtained a reduction *à la* Marsden–Weinstein–Meyer for hybrid Hamiltonian systems.
- Our method does not require the momentum map to be equivariant, nor to be preserved by the impact map.
- This is a first step for developing a reduction by stages for hybrid Hamiltonian systems.

Main references

[1] L. Colombo, M. de León, M. E. Eyrea Irazú, and A. López-Gordón. "Generalized Hybrid Momentum Maps and Reduction by Symmetries of Simple Hybrid Forced Mechanical Systems". *Journal of Mathematical Physics*, **66**(6) (2025).

[2] J. E. Marsden, G. Misiolek, J.-P. Ortega, M. Perlmutter, and T. Ratiu. *Hamiltonian Reduction by Stages*. Lecture Notes in Mathematics. Springer-Verlag: Berlin; Heidelberg, 2007.

[3] A. van der Schaft and Schumacher, Hans. *An Introduction to Hybrid Dynamical Systems*. Vol. 251. Springer: London, 2000.

[4] E. R. Westervelt, J. W. Grizzle, C. Chevallereau, J. H. Choi, and B. Morris. *Feedback Control of Dynamic Bipedal Robot Locomotion*. CRC Press: Boca Raton, 2018.

Merci pour votre attention!

✉ Feel free to contact me at alopez-gordon@impan.pl

🌐 These slides are available at www.alopezgordon.xyz