Reduction, Hamilton-Jacobi theory and discretization of mechanical systems with external forces

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Outline of the presentation

Motivation

External forces appear in many dynamical systems:

- systems with dissipation or friction,
- control forces,
- nonholonomic Čaplygin systems.

Symplectic structure on TQ induced by the Lagrangian

- Let Q be an n-dimensional differentiable manifold with local coordinates (q^i).
- The **vertical endomorphism** $S : T(TQ) \to T(TQ)$ is given by

$$S = \frac{\partial}{\partial \dot{q}^i} \otimes \mathrm{d}q^i.$$

- Consider a Lagrangian function L on TQ.
- The Poincaré-Cartan forms are given by

$$\theta_L = S^*(\mathrm{d}L), \qquad \omega_L = -\mathrm{d}\theta_L.$$

- Hereinafter, L will be assumed to be regular, i.e., ω_L is symplectic.

SODEs

- A **second order differential equation** (**SODE**) is locally of the form

$$\xi = \dot{q}^i \frac{\partial}{\partial q^i} + \xi^i(q^i, \dot{q}^i) \frac{\partial}{\partial \dot{q}^i}.$$

- Clearly, ξ is a SODE if and only if

$$S(\xi) = \Delta,$$

where $\Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}$ is the Liouville vector field.

- A **solution** of a SODE ξ is a curve $\sigma(t) = (q^i(t))$ on Q such that its canonical lift to TQ is an integral curve of ξ, given by

$$\frac{\mathrm{d}^2 q^i}{\mathrm{d}t^2} = \xi^i \left(q^i, \frac{\mathrm{d}q^i}{\mathrm{d}t} \right), \quad 1 \leq i \leq n.$$

Forced Euler-Lagrange equations

- An external force is represented by a semibasic 1-form α on TQ, i.e., $\alpha(Z) = 0$ for any vertical vector field Z on TQ.
- Locally,

$$\alpha = \alpha_i(q, \dot{q})\, \mathrm{d}q^i .$$

- The dynamics is determined by the **forced Euler-Lagrange vector field** $\xi_{L,\alpha}$, given by

$$\iota_{\xi_{L,\alpha}} \omega_L = \mathrm{d}E_L + \alpha,$$

  where $E_L = \Delta(L) - L$.
- $\xi_{L,\alpha}$ is a SODE, with solutions given by the **forced Euler-Lagrange equations**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = -\alpha_i, \quad 1 \leq i \leq n.$$

Vertical and complete lifts of a vector field

- Consider a vector field X on Q locally given by

$$X = X^i \frac{\partial}{\partial q^i}.$$

- Its **vertical lift** is the vector field X^v on TQ given by

$$X^v = X^i \frac{\partial}{\partial \dot{q}^i}.$$

- Its **complete lift** is the vector field X^c on TQ given by

$$X^c = X^i \frac{\partial}{\partial q^i} + \dot{q}^j \frac{\partial X^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i}.$$

Rayleigh forces

- An **Rayleigh force** is an external force of the form

$$\bar{R} = S^*(\mathrm{d}\mathcal{R}) = \frac{\partial \mathcal{R}}{\partial \dot{q}^i} \, \mathrm{d}q^i,$$

  where $\mathcal{R} : TQ \to \mathbb{R}$ is the **Rayleigh potential** or **Rayleigh dissipation function**.
- $\mathcal{R}$ expresses the energy dissipated away by the system:

$$\frac{\mathrm{d}}{\mathrm{d}t} E_L \circ \sigma(t) = -\Delta(\mathcal{R}) \circ \sigma(t),$$

  with σ an integral curve of $\xi_{L,\bar{R}}$.
- Rayleigh considered only forces linear in the velocities, namely,

$$\mathcal{R} = \frac{1}{2} R_{ij}(q) \dot{q}^i \dot{q}^j.$$

Dissipative bracket

Definition

The **dissipative bracket** of a pair of functions f and g on (TQ, ω_L) is given by

$$[f, g] \coloneqq (SX_f)(g) = \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right)^{-1} \frac{\partial f}{\partial \dot{q}^i} \frac{\partial g}{\partial \dot{q}^j}.$$

- It is bilinear and symmetric
- It satisfies the Leibniz rule:

$$[fg, h] = [f, h]g + f[g, h]$$

- f is a constant of the motion of $(L, \mathcal{R})$ iff

$$\{f, E_L\} - [f, \mathcal{R}] = 0.$$

Noether theorem

Theorem (Noether's theorem for forced Lagrangian systems)

Let X be a vector field on Q. Then $X^c(L) = \alpha(X^c)$ if and only if $X^v(L)$ is a constant of the motion.

- A vector field X on Q satisfying these conditions is called a **symmetry of the forced Lagrangian** (L, α).
- For a Rayleigh system $(L, \mathcal{R})$, this is equivalent to

$$X^c(L) = X^v(\mathcal{R}).$$

Example (Fluid resistance)

- Consider a body of mass m moving along 1 dimension through a fluid that fully encloses it.
- The Rayleigh potential associated to the drag force is

$$\mathcal{R} = \frac{k}{3}\dot{q}^3, \qquad k = \frac{1}{2}CA\rho; \qquad L = \frac{1}{2}m\dot{q}^2.$$

- Consider the vector field

$$X = e^{kq/m}\frac{\partial}{\partial q}.$$

- $X^c(L) = X^v(\mathcal{R}) \Longrightarrow X^v(L) = me^{kq/m}\dot{q}$ is a constant of the motion.
- When $k = 0$, X is the generator of translations and the conservation of momentum is recovered.

Other point-like symmetries I

- A **Lie symmetry** is a vector field X on Q such that

$$[X^c, \xi_{L,\alpha}] = \mathcal{L}_{X^c}\xi_{L,\alpha} = 0$$

- If $\mathcal{L}_{X^c}\theta_L$ is closed, then X is a Lie symmetry if and only if

$$\mathcal{L}_{X^c}\alpha = -\mathrm{d}(X^c(E_L)).$$

- A **Noether symmetry** is a vector field X on Q such that

$$\mathcal{L}_{X^c}\theta_L = \mathrm{d}f, \qquad X^c(E_L) + \alpha(X^c) = 0.$$

- If $\mathcal{L}_{X^c}\theta_L = \mathrm{d}f$, then X is a Noether symmetry if and only if $f - X^v(L)$ is a conserved quantity.

Other point-like symmetries II

- For a Rayleigh system $(L, \mathcal{R})$, if $\mathcal{L}_{X^c}\theta_L = \mathrm{d}f$, then X is a Noether symmetry if and only if

$$X^c(E_L) + X^v(\mathcal{R}) = 0.$$

- If X is a Noether symmetry, it is also a symmetry of the forced Lagrangian if and only if $\mathcal{L}_{X^c}\theta_L = 0$.
- If X is a Noether symmetry, it is also a Lie symmetry if and only if

$$\iota_{X^c}\mathrm{d}\alpha = 0.$$

Non-point-like symmetries I

- A vector field $\tilde{X}$ on TQ is called a **dynamical symmetry** if

$$[\tilde{X}, \xi_{L,\alpha}] = 0.$$

- A vector field $\tilde{X}$ on TQ is called a **Cartan symmetry** if

$$\mathcal{L}_{\tilde{X}} \theta_L = \mathrm{d} f, \qquad \tilde{X}(E_L) + \alpha(\tilde{X}) = 0$$

- X is a Lie symmetry if and only if X^c is a dynamical symmetry.
- X is a Noether symmetry if and only if X^c is a Cartan symmetry.

Non-point-like symmetries II

- If $\mathcal{L}_{\tilde{X}}\theta_L$ is closed, then $\tilde{X}$ is a dynamical symmetry if and only if

$$\mathrm{d}(\tilde{X}(E_L)) = -\mathcal{L}_{\tilde{X}}\alpha.$$

- A Cartan symmetry is a dynamical symmetry if and only if

$$\iota_{\tilde{X}}\mathrm{d}\alpha = 0.$$

- If $\mathcal{L}_{\tilde{X}}\theta_L = \mathrm{d}f$, then $\tilde{X}$ is a Cartan symmetry if and only if $f - (S\tilde{X})(L)$ is a constant of the motion.
- For a Rayleigh system $(L, \mathcal{R})$, $\tilde{X}$ is a Cartan symmetry if and only if

$$\tilde{X}(E_L) + (S\tilde{X})(\mathcal{R}) = 0.$$

Momentum map

- Consider a Lie group action of G on Q and the lifted action on TQ.
- Assume the G-action to be free and proper.
- Consider a G-invariant regular Lagrangian L on TQ.
- The **natural momentum map** is given by

$$J : TQ \to \mathfrak{g}^*$$
$$\langle J(x), \xi \rangle = \theta_L(\xi_Q^c)$$

  for each $\xi \in \mathfrak{g}$.
- For each $\xi \in \mathfrak{g}$, we can introduce a function on TQ:

$$J^\xi : TQ \to \mathbb{R}$$
$$x \mapsto \langle J(x), \xi \rangle$$

Lemma

Consider a forced Lagrangian system (L, α). Let $\xi \in \mathfrak{g}$. Then

1. *J^ξ is a conserved quantity if and only if*

$$\alpha(\xi_Q^c) = 0.$$

2. *If the previous equation holds, then ξ leaves α invariant if and only if*

$$\iota_{\xi_Q^c} \mathrm{d}\alpha = 0.$$

In addition, the vector subspace of $\mathfrak{g}$ given by

$$\mathfrak{g}_\alpha = \left\{ \xi \in \mathfrak{g} \mid \alpha(\xi_Q^c) = 0,\ \iota_{\xi_Q^c} \mathrm{d}\alpha = 0 \right\}$$

is a Lie subalgebra of $\mathfrak{g}$.

Theorem

Consider a $\mathfrak{g}_\alpha$-invariant forced Lagrangian system (L, α) on TQ. Let $\mu \in \mathfrak{g}_\alpha^*$. Then:

1. *The quotient space $(TQ)_\mu := J_\alpha^{-1}(\mu)/(G_\alpha)_\mu$ is endowed with an induced symplectic structure ω_μ, given by*

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega_L,$$

where $\pi_\mu : J_\alpha^{-1}(\mu) \to (TQ)_\mu$ and $i_\mu : J_\alpha^{-1}(\mu) \hookrightarrow TQ$.

2. *The reduced Lagrangian L_μ is given by*

$$L_\mu \circ \pi_\mu = L \circ i_\mu.$$

3. *The reduced external force α_μ is given by*

$$\pi_\mu^* \alpha_\mu = i_\mu^* \alpha.$$

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Forced Hamilton equations

- As it is well-known, T^*Q is endowed with a canonical symplectic form $\omega_Q = -\mathrm{d}\theta_Q$, where $\theta_Q = p_i \mathrm{d}q^i$ in Darboux coordinates.
- A **forced Hamiltonian system** is a pair (H, β), where β is a semibasic 1-form on T^*Q.
- The forced dynamical vector field $X_{H,\beta}$ is given by

$$\iota_{X_{H,\beta}} \omega_Q = \mathrm{d}H + \beta.$$

- Its integral curves satisfy the **forced Hamilton equations**:

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i},$$
$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i} - \beta_i.$$

Standard Hamilton-Jacobi problem

- The Hamilton-Jacobi problem consists in finding a **generating function** S on Q such that

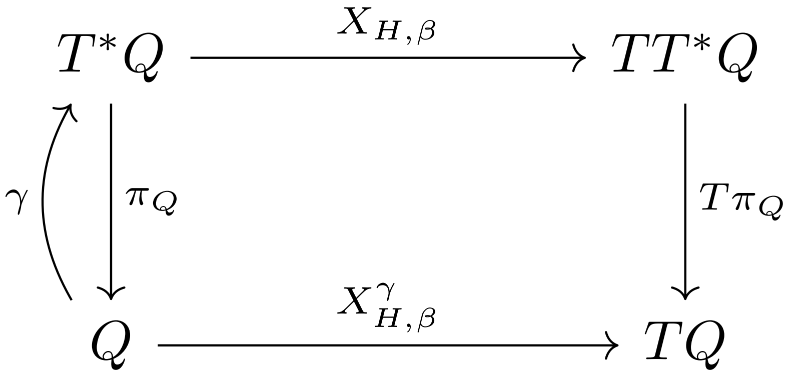
$$H\left(q^i, \frac{\partial S}{\partial q^i}\right) = E.$$

- Geometrically, this equation can be written as

$$\mathrm{d}\left(H \circ \gamma\right) = 0,$$

with $\gamma = \mathrm{d}S$ a section of $\pi_Q : T^*Q \to Q$.

Geometric Hamilton-Jacobi problem



Hamilton-Jacobi problem for (H, β)

Theorem

Let γ be a closed 1-form on Q. Then the following conditions are equivalent:

1. $\mathrm{d}(H \circ \gamma) = -\gamma^* \beta$,
2. *if $\sigma : \mathbb{R} \to Q$ is an integral curve of $X^{\gamma}_{H,\beta}$, then $\gamma \circ \sigma$ is an integral curve of $X_{H,\beta}$;*
3. $\operatorname{Im} \gamma$ *is a Lagrangian submanifold of T^*Q and $X_{H,\beta}$ is tangent to it.*

If γ satisfies these conditions, it is called a solution of the Hamilton-Jacobi problem for (H, β).

Complete solutions I

- A map $\Phi : Q \times \mathbb{R}^n \to T^*Q$ is called **complete solution of the Hamilton-Jacobi problem** for (H, β) if
  1. Φ is a diffeomorphism,
  2. for any $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, the map

$$\begin{aligned}\Phi_\lambda : Q &\to T^*Q\\ q &\mapsto \Phi_\lambda(q) = \Phi(q, \lambda_1, \dots, \lambda_n)\end{aligned}$$

     is a solution of the Hamilton-Jacobi problem for (H, β).

Complete solutions II

- Consider the functions given by

$$f_a = \pi_a \circ \Phi^{-1} : T^*Q \to \mathbb{R},$$

where π_a denotes the projection over the a-th component of $\mathbb{R}^n$.

- The functions f_a are constants of the motion. Moreover, they are in involution, i.e.,

$$\{f_a, f_b\} = 0$$

Example

Consider a n-dimensional forced Hamiltonian system (H, β), with

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2, \qquad \beta = \sum_{i=1}^{n} \kappa_i p_i^2 \mathrm{d}q_i.$$

The functions

$$f_a = e^{\kappa_a q^a} p_a, \qquad a = 1, \ldots, n.$$

are constants of the motion in involution. The 1-form γ on Q given by

$$\gamma = \sum_{i=1}^{n} \lambda_i e^{-\kappa_i q^i} \mathrm{d}q^i$$

is a complete solution of the Hamilton-Jacobi problem.

Reduction and reconstruction of the Hamilton-Jacobi problem

- Let (H, β) be a forced Hamiltonian system on T^*Q.
- Let G be a Lie group that acts freely and properly on Q, and on T^*Q by the cotangent lift action.
- Suppose that this action preserves H and β.
- Then, we can introduce a reduced Hamiltonian $\tilde{H}$ and a reduced external force $\tilde{\beta}$ on $T^*(Q/G)$.
- If γ is a G-invariant solution of the Hamilton-Jacobi problem for (H, β), then it induces a solution $\tilde{\gamma}$ of the Hamilton-Jacobi problem for $(\tilde{H}, \tilde{\beta})$.
- Conversely, we can reconstruct γ from $\tilde{\gamma}$.

Example (Calogero-Moser system with a linear Rayleigh force)

- Consider a forced Hamiltonian system $(H, \tilde{R})$ on $T^*\mathbb{R}^2$, where

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + \frac{1}{(x-y)^2}\right), \qquad \tilde{R} = (p_x + p_y)(\mathrm{d}x - \mathrm{d}y).$$

- Consider the action $\Phi\left(t, (x, y)\right) = (t + x, t + y)$ of $\mathbb{R}$ on $\mathbb{R}^2$.
- Clearly, $(H, \tilde{R})$ is invariant under Φ^{T^*}. The momentum map is $J(x, y, p_x, p_y) = p_x + p_y$.
- We can identify $J^{-1}(\mu)/\mathbb{R}$ with $\mathbb{R}^2$, with coordinates (q, p) and the natural projection $\pi : (x, y, p, \mu - p) \mapsto (x - y, p)$.
- $\tilde{\gamma}_\lambda = \mathrm{d}\tilde{S}_\lambda \rightsquigarrow \gamma_\lambda = \mathrm{d}S_\lambda$, where the generating functions are

$$\tilde{S}_\lambda(q) = \frac{1}{2}q^2 - \frac{1}{2\mu q} + \lambda q, \qquad S_\lambda(x, y) = \tilde{S}_\lambda(x - y) + \mu y.$$

Čaplygin systems

- A **Čaplygin system** is a nonholonomic mechanical system such that:
  1. Q is a fibred manifold, say $\rho : Q \longrightarrow N$, over a manifold N;
  2. the constraints are provided by the horizontal distribution of an Ehresmann connection Γ in ρ;
  3. the Lagrangian $L : TQ \longrightarrow \mathbb{R}$ is Γ-invariant.
- Take coordinates (q^a, q^i) on Q such that $\rho(q^a, q^i) = (q^a)$.
- Let (L, Γ) be a Čaplygin system on TQ. Let $\mathfrak{R}$ be the curvature of Γ. Then,

$$\ell(q^a, \dot{q}^a) = L\left(q^a, q^i, \dot{q}^a, -\Gamma^i_a \dot{q}^a\right), \qquad \alpha = \left(\frac{\partial L}{\partial \dot{q}^i} \dot{q}^b \mathfrak{R}^i_{ab}\right) \mathrm{d}q^b,$$

is a forced Lagrangian system on TN equivalent to (L, Γ).

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Lagrange-D'Alembert principle

- The dynamics $q(t)$ of the forced Lagrangian system (L, α) can be obtained from

$$\delta \int_0^T L(q(t), \dot{q}(t)) \, \mathrm{d}t - \int_0^T \alpha(q(t), \dot{q}(t)) \cdot \delta q(t) \, \mathrm{d}t = 0,$$

  where δ denotes variations vanishing at the endpoints.
- This leads to the forced Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = -\alpha_i, \quad 1 \leq i \leq n.$$

Discrete Lagrangian mechanics I

- The continuous objects are now replaced by their discrete counterparts:

$$
\begin{aligned}
&TQ \rightsquigarrow Q \times Q \\
&L \rightsquigarrow L_d : Q \times Q \to \mathbb{R} \\
&\alpha \rightsquigarrow f_d = \left(f_d^+, f_d^-\right) \in \Omega^1(Q \times Q) \\
&q(t) \rightsquigarrow \{q_k\}_{k=0}^N \in Q^{N+1}
\end{aligned}
$$

Discrete Lagrangian mechanics II

- The dynamics is given by the **discrete Lagrange-d'Alembert principle**:

$$\delta \sum_{k=0}^{N-1} L_d\left(q_k, q_{k+1}\right) + \sum_{k=0}^{N-1} \left[f_d^-\left(q_k, q_{k+1}\right) \delta q_k + f_d^+\left(q_k, q_{k+1}\right) \delta q_{k+1}\right] = 0,$$

  for all variations δ_k vanishing at the endpoints.

- It leads to the **forced discrete Euler-Lagrange equations**:

$$D_2 L_d\left(q_{k-1}, q_k\right) + D_1 L_d\left(q_k, q_{k+1}\right) + f_d^+\left(q_{k-1}, q_k\right) + f_d^-\left(q_k, q_{k+1}\right) = 0.$$

Discrete Lagrangian mechanics III

- The **exact** discrete Lagrangian and external forces are

$$L_d^{\mathrm{ex}}(q_j, q_{j+1}) = \int_{t_j}^{t_{j+1}} L(q(t), \dot{q}(t))\, \mathrm{d}t,$$

$$f_d^{E+}(q_j, q_{j+1}) = -\int_{t_j}^{t_{j+1}} \alpha(q(t), \dot{q}(t)) \cdot \frac{\partial q(t)}{\partial q_{j+1}}\, \mathrm{d}t,$$

$$f_d^{E-}(q_j, q_{j+1}) = -\int_{t_j}^{t_{j+1}} \alpha(q(t), \dot{q}(t)) \cdot \frac{\partial q(t)}{\partial q_j}\, \mathrm{d}t.$$

Discrete Lagrangian mechanics IV

- In practice, one takes an approximation of the integrals above.

Example (Midpoint rule)

Let $L = L(q, \dot{q})$ and $\alpha = \alpha(q, \dot{q})$ be a forced continuous Lagrangian system. Then,

$$L_d^{\frac{1}{2}}(q_0, q_1) = h\ L\left(\frac{q_0 + q_1}{2}, \frac{q_1 - q_0}{h}\right),$$
$$f_d^{\frac{1}{2}+}(q_0, q_1) = f_d^{\frac{1}{2}-}(q_0, q_1) = -h\ \alpha\left(\frac{q_0 + q_1}{2}, \frac{q_1 - q_0}{h}\right),$$

where h is a fixed time step.

Discrete Rayleigh forces

Definition

A discrete force $f_d = (f_d^-, f_d^+)$ is **Rayleigh** if there exists a function $\mathcal{R}_d$ on $Q \times Q$ such that

$$f_d^-(q_0, q_1) = D_1 \mathcal{R}_d(q_0, q_1), \qquad f_d^+(q_0, q_1) = -D_2 \mathcal{R}_d(q_0, q_1).$$

$\mathcal{R}_d$ is called the **discrete Rayleigh potential**.

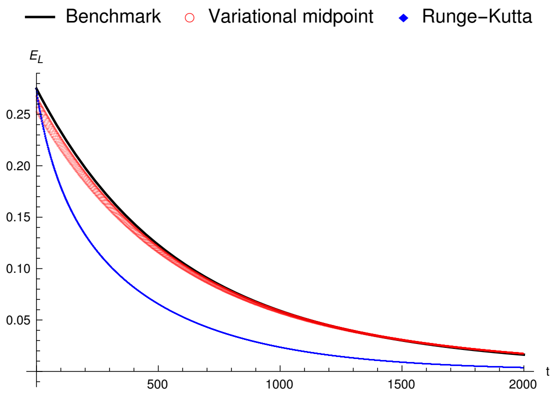
Example (midpoint rule)

Suppose that $\mathcal{R}$ is a homogeneous Rayleigh potential, i.e., $\mathcal{R} = \mathcal{R}(\dot{q})$. Then,

$$\mathcal{R}_d^{\frac{1}{2}}(q_0, q_1) = \frac{h}{2} \mathcal{R}\left(\dot{q} = \frac{q_1 - q_0}{h}\right)$$

is a discrete Rayleigh potential.

Variational integrators ⋙ classical numerical integrators



$$L = \frac{1}{2}\,||\dot{q}||^2 - ||q||^2 \left(||q||^2 - 1\right)^2, \qquad \mathcal{R} = \frac{1}{2} k\,||\dot{q}||^2$$

Discrete Hamilton equations I

- The **forced discrete Legendre transforms** define the following momenta:

$$p_{j+1} = D_2 L_d\left(q_j, q_{j+1}\right) + f_d^+\left(q_j, q_{j+1}\right),$$
$$p_j = -D_1 L_d\left(q_j, q_{j+1}\right) - f_d^-\left(q_j, q_{j+1}\right).$$

- We can define the **right discrete Hamiltonian**:

$$H_d^+\left(q_j, p_{j+1}\right) = p_{j+1} q_{j+1} - L_d\left(q_j, q_{j+1}\right),$$

- The **discrete action** is

$$S_d^N\left(\{q_j\}\right) = \sum_{j=0}^{N-1} L_d\left(q_j, q_{j+1}\right) = \sum_{j=0}^{N-1}\left[p_{j+1} q_{j+1} - H_d^+\left(q_j, p_{j+1}\right)\right]$$

Discrete Hamilton equations II

- From the discrete Lagrange-d'Alembert principle, one can derive the **forced right discrete Hamilton equations**:

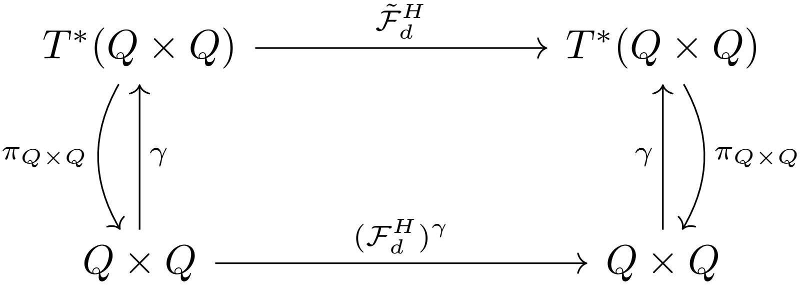
$$\left[q_{j+1} - D_2 H_d^+ \left(q_j, p_{j+1}\right)\right] \frac{\partial p_{j+1}}{\partial q_{j+1}} = -f_d^+(q_j, q_{j+1}),$$
$$p_j = D_1 H_d^+ \left(q_j, p_{j+1}\right) - f_d^- \left(q_j, q_{j+1}\right).$$

Forced discrete Hamilton-Jacobi theory

- Consider the discrete flow $\mathcal{F}_d^H : (q_j, p_j) \mapsto (q_{j+1}, p_{j+1})$. Let

$$\tilde{\mathcal{F}}_d^H : T^*(Q \times Q) \to T^*(Q \times Q)$$
$$(q_{j-1}, q_j, p_{j-1}, p_j) \mapsto (q_j, q_{j+1}, p_j, p_{j+1}) \, .$$

- Idea: define a section γ on $T^*(Q \times Q)$ and a discrete flow $\left(\mathcal{F}_d^H\right)^\gamma : Q \times Q \to Q \times Q$ such that the following diagram commutes:


$T^*(Q \times Q)$
$\tilde{\mathcal{F}}_d^H$
$T^*(Q \times Q)$
$\pi_{Q \times Q}$
γ
γ
$\pi_{Q \times Q}$
$Q \times Q$
$(\mathcal{F}_d^H)^\gamma$
$Q \times Q$

$$
\begin{array}{ccc}
(q_{j-1}, q_j, p_{j-1}, p_j) & \xrightarrow{\quad \tilde{\mathcal{F}}_d^H \quad} & (q_j, q_{j+1}, p_j, p_{j+1}) \\
\pi_{Q\times Q} \downarrow \; \uparrow \gamma & & \gamma \uparrow \; \downarrow \pi_{Q\times Q} \\
(q_{j-1}, q_j) & \xrightarrow{\quad (\mathcal{F}_d^H)^\gamma \quad} & (q_j, q_{j+1})
\end{array}
$$

Forced discrete Hamilton-Jacobi theorem I

- Let us introduce the following mappings

$$\begin{aligned}\gamma^{+} &:= DS_d \circ \pi_2 + f_d^{+} : Q \times Q \to T^*Q \\ &\qquad (q_j, q_{j+1}) \mapsto (q_{j+1}, p_{j+1}), \\ \mathcal{F}^{+} &: Q \times Q \to Q \\ \mathcal{F}^{+}(q_{j-1}, q_j) &:= D_2 H_d^{+}\left(q_j, \gamma^{+}(q_j, \mathcal{F}^{+}(q_{j-1}, q_j))\right) \\ &\quad - f_d^{+}\left(q_j, \mathcal{F}^{+}(q_{j-1}, q_j)\right) \left[D_2\gamma^{+}(q_j, \mathcal{F}^{+}(q_{j-1}, q_j))\right]^{-1}\end{aligned}$$

Forced discrete Hamilton-Jacobi theorem II

Theorem

Suppose that

1. S_d *and* γ^+ *satisfy the* ***forced right discrete H-J equation****:*

$$S_d^{j+1}(q_{j+1}) - S_d^j(q_j) - \gamma^+(q_j, q_{j+1})q_{j+1} + H_d^+\left(q_j, \gamma^+(q_j, q_{j+1})\right) = 0,$$

2. *the sequence of points* $\{c_k\}_{k=0}^N \subset Q$ *satisfies*

$$c_{k+1} = \mathcal{F}^+(c_{k-1}, c_k).$$

Then, the set of points $\{(c_k, p_k)\}_{k=0}^N \subset T^*Q$ *with*

$$p_{k+1} = \gamma^+(q_{k-1}, q_k)$$

is a solution of the forced right discrete Hamilton equations.

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Thank you!