Homogeneous Darboux and Frobenius theorems on graded manifolds

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The interest of gradings

There are several scenarios in geometry and physics in which a $(\mathbb{N}, \mathbb{Z}, \mathbb{Z}_2, \mathbb{R}\ \ldots)$ grading appears:

- the algebra of exterior forms with the exterior product $\left(\Omega^\bullet(M), \wedge\right)$,
- the spin of particles,
- intensive/extensive variables in thermodynamics,
- symplectisation/Poissonisation of contact/Jacobi manifolds,
- supermanifolds,
- higher tangent bundles.

Theorem (Euler)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, and k an integer. The following assertions are equivalent:

1. $f(t \cdot x) = t^k f(x)\,, \quad \forall\, t \in \mathbb{R} \setminus \{0\}\,,\ \forall\, x \in \mathbb{R}^n\,,$
2. *f is an eigenfunction of $X = \sum_{i=1}^{n} x^i \partial_{x^i}$ with eigenvalue k, namely*

$$X(f) = k \cdot f\,.$$

Definition

A function f satisfying any of the equivalent conditions above is called **homogeneous of degree** k or k-**homogeneous**.

We can extend this notion to a manifold M^n by considering a vector field $X \in \mathfrak{X}(M)$ which is locally of the form

$$X = \sum_{i=1}^{n} x^i \partial_{x^i}$$

in a certain atlas.

Definition

An (even) vector field ∇ on a (super)manifold M is called a **weight vector field** if in a neighbourhood of every point of (the body of) M there are coordinates (x^a) such that

$$\nabla = \sum_{a=1}^{n} w_a \cdot x^a \partial_{x^a} , \quad w_a \in \mathbb{R} .$$

Such coordinates are called **homogeneous coordinates**, and the pair (M, ∇) is called a **homogeneity (super)manifold**.

Definition

Let (M, ∇) be a homogeneity (super)manifold and $w \in \mathbb{R}$. A tensor field A on M is called **homogeneous of degree** w or w-**homogeneous** if

$$\mathcal{L}_{\nabla} A = w \cdot A \,.$$

Example (Trivial)

The zero-section of the tangent bundle makes any (super)manifold a homogeneity (super)manifold:

$$\nabla \equiv 0\,.$$

This means that all the subsequent results I shall present still hold if you forget the adjective “homogeneous”.

Example (Vector bundles)

Let $\pi\colon E \to M$ be a vector bundle (VB). The Euler vector field $\nabla_E \in \mathfrak{X}(E)$, i.e. the infinitesimal generator of homotheties on the fibers, is a weight vector field. In bundle coordinates,

$$\pi\colon (x^i, y^a) \mapsto (x^i)\,, \quad \nabla_E = \sum_a y^a \partial_{y^a}\,.$$

Remark

The structure of VB on E is uniquely determined[a] by its structure of manifold and a smooth action of the monoid $(\mathbb{R}, \cdot)$ generated by ∇_E.

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[a] See Grabowski and Rotkiewicz, *J. Geom. Phys.* **59** (2009).

Example (Exact symplectic manifolds)

Let (M, ω) be a symplectic manifold. Then, the following statements are equivalent:

1. ω is exact, i.e. there exists a $\theta \in \Omega^1(M)$ such that $\omega = \mathrm{d}\theta$,
2. there exists a **Liouville vector field** $\nabla \in \mathfrak{X}(M)$ such that $\mathcal{L}_\nabla \omega = \omega$.

In fact, since $\mathfrak{X}(M) \ni X \mapsto \iota_X \omega \in \Omega^1(M)$ is an isomorphism, given θ (resp. ∇), we can univocally define θ (resp. ∇) by the relation

$$\iota_\nabla \omega = \theta\,.$$

The Liouville vector field is a weight vector field. Indeed, in Darboux coordinates (q^i, p_i) for θ, we have

$$\theta = p_i \mathrm{d}q^i \Longrightarrow \nabla = p_i \partial_{p_i}\,.$$

Given a homogeneity manifold (M, ∇) and an open subset $U \subseteq M$, it is important to distinguish two possible situations:

1. $\nabla|_U$ is nowhere zero,
2. there exists a point $x_0 \in M$ such that $\nabla(x_0) = 0$.

Proposition

Any nowhere-vanishing vector field X on a manifold M^n is a weight vector field. However, its weights are not canonical.

Proof.

Since X is nowhere zero, there exists an atlas with local coordinates (x^a) such that $X = \partial_{x^1}$. For any $\Gamma := \{w_1, \dots, w_n\} \subset \mathbb{R}$ with $w_1 \neq 0$, we can define a new system of coordinates

$$y^1 = e^{w_1 x^1}\,, \quad y^i = e^{w_i x^1} x^i\,, \quad 2 \leq i \leq n\,,$$

so that

$$X = \sum_{a=1}^{n} w_a \cdot y^a \partial_{y^a}\,.$$

□

On the other hand, in a neighbourhood of any point at which the weight vector field vanishes, its weights are canonical.

Proposition (Grabowska and Grabowski, 2024)

An even vector field $\nabla \in \mathfrak{X}(M)$ *is a weight vector field iff* ∇ *is locally linear and* $\mathsf{T}_{x_0}\nabla$ *is diagonalizable at any* $x_0 \in |M|$ *such that* $\nabla(x_0) = 0$*. In the case* $\nabla(x_0) = 0$*, all weights of systems of homogeneous coordinates around* x_0 *are the same for each system of homogeneous coordinates up to permutations among the weights of even and odd coordinates separately.*

A review on superanalysis

- The superspace $\mathbb{R}^{p|q}$ has canonical coordinates $(x^1, \dots, x^p, \xi^1, \dots, \xi^q)$, where x are commuting and ξ anti-commuting:

$$x^i \cdot x^j = x^j \cdot x^i \,, \quad x^i \cdot \xi^a = \xi^a \cdot x^i \,, \quad \xi^a \cdot \xi^b = -\xi^b \cdot \xi^a \,.$$

- Smooth functions on $\mathbb{R}^{p|q}$ are polynomials on the anticommuting variables ξ with functions on the commuting variables x as coefficients, e.g. in $\mathbb{R}^{p|2}$ these are of the form

$$f(x^1, \dots x^p, \xi^1, \xi^2) = f_0(x) + f_1(x)\,\xi^1 + f_2(x)\,\xi^2 + f_{12}(x)\,\xi^1 \cdot \xi^2 \,.$$

- The fact that there are commuting and anti-commuting coordinates makes supermanifolds equipped with a $\mathbb{Z}_2$-grading.
- We call objects with $\mathbb{Z}_2$-degree 0 (resp. 1) **even** (resp. **odd**).

A review on superanalysis

- A tangent vector v at a point $p \in \mathbb{R}^{p|q}$ is defined as a superderivation on the space of functions:

  $$v(f \cdot g) = v(f) \cdot g(p) + (-1)^{|v|\,|f|} f(p) \cdot v(g)\,,$$

  where $|\cdot|$ denotes the $\mathbb{Z}_2$-grading.
- Coordinates (x^i, ξ^a) induce a basis $(\partial_{x^i}, \partial_{\xi^a})$ of $\mathrm{T}_p\mathbb{R}^{p|q}$ such that

  $$\partial_{x^i}(x^j) = \delta_i^j\,, \quad \partial_{\xi^a}(\xi^b) = \delta_a^b\,, \quad \partial_{x^i}(\xi^a) = 0 = \partial_{\xi^a}(x^i)\,.$$
- With this, it is possible to extend the notions of vector field, differential form, (co)tangent bundle, and (co)distribution.
- The wedge product now depends on the $\mathbb{Z}_2$-grading, as well as the usual $\mathbb{N}$-grading:

  $$\alpha \wedge \beta = (-1)^{kl + |\alpha|\,|\beta|} \beta \wedge \alpha\,,$$

  for any k-form α and any l-form β.

Notions of supermanifold

- In the literature, a supermanifold $\mathcal{M}$ is defined either
    - as a set endowed with compatible smooth charts of (super)coordinates taking values in an exterior algebra $\bigwedge^\bullet(V)$, or
    - as an ordinary manifold $M_0 = |\mathcal{M}|$ equipped with a sheaf of functions which is locally isomorphic to $\mathscr{C}^\infty(M_0) \otimes \bigwedge^\bullet(V)$.
- Both definitions are actually interchangeable, namely, there is an equivalence of categories.
- In particular, there is an isomorphism of sheaves which permits identifying systems of coordinates on both approaches.

Frobenius theorem

Definition

A distribution $D \subseteq \mathrm{T}M$ (resp. codistribution $D \subseteq \mathrm{T}^*M$) on a homogeneity (super)manifold (M, ∇) is called a **homogeneous distribution** if the tangent lift $\mathrm{d}_{\mathrm{T}}\nabla$ (resp. the cotangent lift $\mathrm{d}_{\mathrm{T}}^*\nabla$) is tangent to D.

Theorem (Grabowska and Grabowski, 2025)

$D \subseteq \mathrm{T}M$ is a homogeneous distribution iff it is locally generated by homogeneous vector fields.

Corollary

$D \subseteq \mathrm{T}^*M$ is a homogeneous codistribution iff it is locally generated by homogeneous one-forms.

Homogeneous Frobenius theorem

Theorem (Grabowska and Grabowski, 2025)

Let (M, ∇) *be a homogeneity (super)manifold, and let* D *be a homogeneous* *involutive* *distribution of rank* k *on* M*. Around every point* $x_0 \in |M|$ *at which* ∇ *vanishes, there exists a system of homogeneous coordinates* (x^a) *such that* D *is locally generated by* $\partial_{x^1}, \ldots, \partial_{x^k}$*.*

Homogeneous Frobenius theorem

Sketch of the proof.

- We can choose homogeneous coordinates (x^a) around x_0 such that $\partial_{x^1}, \ldots, \partial_{x^k}$ is a basis of D_{x_0}.
- Then, D is locally spanned by sections

$$X_i = \partial_{x^i} + \sum_{j=k+1}^{n} f_i^j \, \partial_{x^j} \,, \quad 1 \leq i \leq k \,.$$

- D is $\mathrm{d}_T \nabla$-invariant $\Longrightarrow$ X_i are homogeneous.
- D involutive $\Longrightarrow$ $[X_i, X_j] = 0$.
- Proceeding as in the proof of the usual Frobenius theorem, we can obtain coordinates straightening X_i.

☐

Remark

If $f \in \mathscr{C}^\infty(M)$ is a w-homogeneous function on a homogeneity manifold (M, ∇) with non-zero weight w, then it vanishes at every point $x_0 \in M$ at which ∇ vanishes. Indeed,

$$f(x_0) = \frac{1}{w}\left(\mathcal{L}_\nabla f\right)(x_0) = \frac{1}{w}\left(\iota_\nabla \mathrm{d}f\right)(x_0) = 0\,.$$

In other words, a homogeneous function non-vanishing at x_0 is necessarily of degree zero.

Consequently, if a homogeneous one-form θ does not vanish at x_0, then $\deg(\theta) \in \Gamma$, with $\Gamma \subset \mathbb{R}$ the set of weights of any system of homogeneous coordinates (x^a) around x_0. Otherwise, all the coefficients of θ in the basis $(\mathrm{d}x^a)$ would vanish at x_0.

Darboux theorems

Homogeneous Poincaré lemma

Lemma (Grabowska and Grabowski, 2024)

Let ω be a λ-homogeneous k-form (with $k > 0$ and $\lambda \in \mathbb{R}$) on a homogeneity (super)manifold (M, ∇). In a neighbourhood of each $x_0 \in M$ such that $\nabla(x_0) = 0$, there exists a $(k-1)$-form α such that:

1. $\mathrm{d}\alpha = \omega$,
2. *α is λ-homogeneous,*
3. $\alpha(x_0) = 0$.

Homogeneous symplectic Darboux theorem

Theorem (Grabowska and Grabowski, 2024)

Let ω be a (λ, σ)-homogeneous symplectic form on a homogeneity (super)manifold (M, ∇) (with $\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{Z}_2$). Around every $x_0 \in |M|$ such that $\nabla(x_0) = 0$, there is a system of homogeneous coordinates (q^i, p_i, ξ^l) such that

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i + \sum_l \varepsilon^l \mathrm{d}\xi^l \wedge \mathrm{d}\xi^l\,, \quad \varepsilon^l = \pm 1\,.$$

Definition

A **presymplectic form** ω on a (super)manifold M is a closed 2-form of constant rank r. Its **characteristic distribution** $C_\omega \subseteq \mathrm{T}M$ is given by

$$C_\omega = \ker \omega\,.$$

Proposition

The characteristic distribution C_ω is an integrable distribution. Moreover, if ω is homogeneous (w.r.t. a weight vector field ∇ on M), then C_ω is a homogeneous distribution.

Homogeneous presymplectic Darboux theorem

Corollary

Let ω *be* (λ, σ)*-homogeneous presymplectic form on a homogeneity (super)manifold* (M, ∇) *(with* $\lambda \in \mathbb{R}$ *and* $\sigma \in \mathbb{Z}_2$*). Around any point* $m \in |M|$ *such that either* $\nabla(m) = 0$ *or* $\nabla(m) \neq 0$ *and* $\nabla(m) \notin \ker \omega_m$*, there is a system of homogeneous coordinates* $(q^i, p_i, \xi^l, z^a, \chi^b)$ *such that*

$$\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q^i + \sum_l \varepsilon^l \mathrm{d}\xi^l \wedge \mathrm{d}\xi^l\,, \quad \varepsilon^l = \pm 1\,.$$

Class of a one-form

Definition

Let α be a k-form on a supermanifold M. The subset

$$\chi(\alpha) = \ker(\alpha) \cap \ker(\mathrm{d}\alpha) \subseteq \mathrm{T}M$$

is called the **characteristic set** of α.
If $\chi(\alpha)$ is a distribution, it is called the **characteristic distribution** of α, we say that α is **regular**, and the corank of $\chi(\alpha)$ as a sub-bundle of $\mathrm{T}M$ is called the **class of** α:

$$\operatorname{class}(\alpha) \coloneqq \operatorname{corank}\left(\chi(\alpha)\right) .$$

Class of a one-form

Remark

For a one-form α on a (standard) manifold M, this is equivalent to the classical definition of class, namely:

- $\operatorname{class}(\alpha) = 2s + 1$ iff $\begin{cases} \alpha \wedge \mathrm{d}\alpha^s \neq 0\,, \\ \mathrm{d}\alpha^{s+1} = 0\,. \end{cases}$
- $\operatorname{class}(\alpha) = 2s$ iff $\begin{cases} \alpha \wedge \mathrm{d}\alpha^{s-1} \neq 0\,, \\ \mathrm{d}\alpha^s \neq 0\,, \\ \alpha \wedge \mathrm{d}\alpha^s = 0\,. \end{cases}$

Class of a one-form

Proposition

If α is a regular form, then $\chi(\alpha)$ is involutive and α is $\chi(\alpha)$-invariant.

Proof.

For any pair of sections X and Y of $\chi(\alpha) = \ker \alpha \cap \ker \mathrm{d}\alpha$,

$$\iota_{[X,Y]}\alpha = [\iota_X, \mathcal{L}_Y]\alpha = \iota_X \iota_Y \mathrm{d}\alpha = 0\,, \quad \iota_{[X,Y]}\mathrm{d}\alpha = 0\,,$$

and

$$\mathcal{L}_X\alpha = \mathrm{d}(\iota_X\alpha) + \iota_X\mathrm{d}\alpha = 0\,.$$

☐

Non-degenerate one-forms

Definition

A regular one-form α on a (super)manifold M is called **non-degenerate** if its characteristic foliation is trivial:

$$\chi(\alpha) = \{0_M\}\,,$$

or equivalently,

$$\operatorname{class}(\alpha) = \dim(M)\,.$$

Non-degenerate one-forms

The annihilator of $\chi(\alpha)$ is given by

$$\big(\chi(\alpha)\big)^\circ = (\ker\alpha \cap \ker \mathrm{d}\alpha)^\circ = (\ker\alpha)^\circ + (\ker \mathrm{d}\alpha)^\circ = \langle\alpha\rangle + \operatorname{Im}(\flat_{\mathrm{d}\alpha})\,,$$

where $\flat_{\mathrm{d}\alpha}\colon \mathrm{T}M \ni v \mapsto \iota_v \mathrm{d}\alpha \in \mathrm{T}^*M$.

The form is non-degenerate iff $\big(\chi(\alpha)\big)^\circ = \mathrm{T}^*M$, so there are two possible cases for $\dim M \geq 2$:

1. $\mathrm{T}^*M = \langle\alpha\rangle \oplus \operatorname{Im}(\flat_{\mathrm{d}\alpha}) \Longrightarrow \operatorname{class}(\alpha) = \operatorname{rank}(\flat_{\mathrm{d}\alpha}) + 1$ (**contact form**),
2. $\mathrm{T}^*M = \operatorname{Im}(\flat_{\mathrm{d}\alpha}) \Longrightarrow \operatorname{class}(\alpha) = \operatorname{rank}(\flat_{\mathrm{d}\alpha})$ (**symplectic potential**).

Non-degenerate one-forms

The situation $\dim M = 1$, on the other hand, is trivial, since then every one-form is closed.

Remark

On a (standard) manifold, the rank of $\mathrm{d}\alpha$ is always even. Thus, a non-degenerate form α is

- a symplectic potential iff $\dim M$ is even,
- contact iff $\dim M$ is odd.

Definition

Let α be a regular one-form on a (super)manifold M. We call α a **precontact form** (resp. a **presymplectic** potential) if the induced one-form α_{red} on $M/\chi(\alpha)$ is a **contact form** (resp. a **symplectic** potential).

Remark

If α is regular, then $\mathrm{d}\alpha$ is presymplectic.

Darboux theorem for homogeneous one-forms

Theorem (Grabowski and L. G., 2025)

Let α be a regular homogeneous one-form of degree $\lambda = (\sigma, w) \in \mathbb{Z}_2 \times \mathbb{R}$ on a homogeneity supermanifold (M, ∇). Around each point $x_0 \in |M|$ such that either $\nabla(m) = 0$ or $\nabla(m) \neq 0$ and $\nabla(m) \notin \chi(\alpha)_m$:

① *For a precontact form α of class $2r + s + 1$, if $\nabla(x_0) = 0$ or $w \neq 0$,*

$$\alpha = \mathrm{d}z + \sum_{i=1}^{r} p_i \mathrm{d}q^i + \sum_{l=1}^{s} \varepsilon^l y^l \mathrm{d}y^l, \quad \varepsilon^l = \pm 1\,,$$

in a certain system of homogeneous coordinates (q^i, p_i, z, y^l, x^a) centered at x_0.

The coordinates (y^l) only appear if α is even, i.e. $\sigma = 0$.

Darboux theorem for homogeneous one-forms

Theorem (Grabowski and L. G., 2025)

Let α be a regular homogeneous one-form of degree $\lambda = (\sigma, w) \in \mathbb{Z}_2 \times \mathbb{R}$ on a homogeneity supermanifold (M, ∇). Around each point $x_0 \in |M|$ such that either $\nabla(m) = 0$ or $\nabla(m) \neq 0$ and $\nabla(m) \notin \chi(\alpha)_m$:

② *If α is a precontact form of class $2r + s + 1$, $\nabla(x_0) \neq 0$ and $w = 0$,*

$$\alpha = \frac{\mathrm{d}z}{z} + \sum_{i=1}^{r} p_i \mathrm{d}q^i + \sum_{l=1}^{s} \varepsilon^l y^l \mathrm{d}y^l, \quad \varepsilon^l = \pm 1\,,$$

in a certain system of homogeneous coordinates (q^i, p_i, z, y^l, x^a) such that

$$z(x_0) = 1\,, \quad q^i(x_0) = p_i(x_0) = y^l(x_0) = x^a(x_0) = 0\,.$$

The coordinates (y^l) only appear if α is even, i.e. $\sigma = 0$.

Darboux theorem for homogeneous one-forms

Theorem (Grabowski and L. G., 2025)

Let α be a regular homogeneous one-form of degree $\lambda = (\sigma, w) \in \mathbb{Z}_2 \times \mathbb{R}$ on a homogeneity supermanifold (M, ∇). Around each point $x_0 \in |M|$ such that either $\nabla(m) = 0$ or $\nabla(m) \neq 0$ and $\nabla(m) \notin \chi(\alpha)_m$:

③ *If α is a presymplectic potential of class $2r + s$,*

$$\alpha = \sum_{i=1}^{r} p_i \mathrm{d}q^i + \sum_{l=1}^{s} \varepsilon^l y^l \mathrm{d}y^l, \quad \varepsilon^l = \pm 1\,,$$

in a certain system of homogeneous coordinates (q^i, p_i, y^l, x^a)

The coordinates (y^l) only appear if α is even, i.e. $\sigma = 0$.

Corollary

Let α be a regular λ-homogeneous one-form on a homogeneity manifold (M, ∇). Around each point $x_0 \in M$ such that either $\nabla(m) = 0$ or $\nabla(m) \neq 0$ and $\nabla(m) \notin \chi(\alpha)_m$, there exists a system of homogeneous coordinates in which α has a canonical expression:

1. $\alpha = \mathrm{d}z + \sum_{i=1}^{r} p_i \mathrm{d}q^i$, *if α is precontact, and* $\nabla(x_0) = 0$ *or* $w \neq 0$,
2. $\alpha = \frac{\mathrm{d}z}{z} + \sum_{i=1}^{r} p_i \mathrm{d}q^i$, $\nabla = z\partial_z$ *and* $z(m) = 1$ *if α is precontact,* $\nabla(x_0) \neq 0$ *and* $w = 0$,
3. $\alpha = \sum_{i} p_i \mathrm{d}x^i$, *if α is a presymplectic potential.*

Structure of the proof

- W.l.o.g. assume α to be non-degenerate. If α is degenerate, then we can obtain the canonical expression for the (local) reduced form α_{red} on $M/\chi(\alpha)$ and pull it back.
- If α is as a contact form, we can express the presymplectic form $\mathrm{d}\alpha$ in homogeneous Darboux coordinates, and then apply the homogeneous Poincaré lemma.
- If α is a presymplectic potential, we construct a homogeneous contact form $\eta = \mathrm{d}t + \alpha$ on $M \times \mathbb{R}$.

Example (Non-trivial homogeneous forms of weight zero)

- Consider $\mathbb{R}^3$ with canonical coordinates (x, y, z) and $\nabla = x\partial_x - y\partial_y$.
- Arbitrary smooth functions $f_i \colon \mathbb{R}^3 \to \mathbb{R}$ of the form

$$f_i(x, y, z) = \varphi_i(xy, z)$$

  are homogeneous of weight 0.
- Let $q = x(1 + f_1)\,, \quad p = y(1 + f_2)\,, \quad \zeta = f_3$ such that

$$\left.\frac{\partial f_3}{\partial z}\right|_{(0,0,0)} \neq 0$$

- Then, $\eta = \mathrm{d}\zeta + p\mathrm{d}q$ is a (local) homogeneous contact form of weight 0.

Example (Non-trivial homogeneous forms of weight zero)

- Consider $\mathbb{R}^3$ with canonical coordinates (x, y, z) and $\nabla = x\partial_x - y\partial_y$.
- For instance,

$$\eta = y\Big(1 + \sin z + \cos(xy)(1 + \sin z) - \sin(xy)\big(e^z + xy(1 + \sin z)\big)\Big)\,\mathrm{d}x \\ - x\sin(xy)\Big(xy(\sin z + 1) + e^z\Big)\,\mathrm{d}y + e^z\cos(xy)\,\mathrm{d}z$$

is a homogeneous contact form of weight 0.

- A system of homogeneous Darboux coordinates is

$$q = x\big(1 + \cos(xy)\big)\,, \quad p = y\big(1 + \sin(xy)\big)\,, \quad \zeta = e^z\cos(xy)\,,$$

so that $\eta = \mathrm{d}\zeta + p\mathrm{d}q$ and $\nabla = q\partial_q - p\partial_p$.

Example (Non-trivial homogeneous forms of weight zero)

- Consider $\mathbb{R}^3$ with canonical coordinates (x, y, z) and $\nabla = x\partial_x - y\partial_y$.
- Similarly, the one-form

$$\begin{aligned}\theta = y\Big(\cosh(xy) + 1\Big)\Big(\sinh(xy) + xy\cosh(xy) + 1\Big)\,\mathrm{d}x\\ + x^2y\cosh(xy)\Big(\cosh(xy) + 1\Big)\,\mathrm{d}y\end{aligned}$$

  is a homogeneous presymplectic potential of weight 0.
- Homogeneous Darboux coordinates (q, p, ζ) are given by

$$q = x\Big(1 + \sinh(xy)\Big)\,, \quad p = y\Big(1 + \cosh(xy)\Big)\,, \quad \zeta = z\,.$$

$\mathbb{N}$-manifolds

Definition (Bursztyn, Cueca and Mehta, 2025)

Let $V = \bigoplus_{i=1}^{n} V_i$ be an $\mathbb{N}$-graded vector space. An (**algebro-geometric**) **$\mathbb{N}$-graded manifold of degree** n is a ringed space $\mathcal{M} = (M, \mathscr{C}_{\mathcal{M}})$ consisting of a smooth manifold M endowed with a sheaf of graded commutative algebras such that any point in M admits a neighborhood U with an isomorphism

$$\mathscr{C}_{\mathcal{M}}|_U \simeq \mathscr{C}^{\infty}_U \otimes \mathrm{S}^{\bullet} V\,, \tag{1}$$

where $\mathrm{S}^{\bullet} V$ denotes the graded symmetric algebra of V. The subsheaf of homogeneous functions of degree l is given by

$$\left.\mathscr{C}^{l}_{\mathcal{M}}\right|_U \simeq \mathscr{C}^{\infty}_U \otimes \mathrm{S}^{l} V\,.$$

The dimension of $\mathcal{M}$ is $m_0 | \cdots | m_n$, where $m_0 = \dim M$ and $m_i = \dim V_i$ for each $i = 1, \ldots, n$.

Definition (Bursztyn, Cueca and Mehta, 2025)

A chart

$$\left(U \subseteq M; x^\alpha, e_i^{\beta_i}\right), \quad 1 \leq \alpha \leq m_0, \quad 1 \leq \beta_i \leq m_i, \quad 1 \leq i \leq n.$$

of $\mathcal{M}$ consists of a chart $(U; x^\alpha)$ of M such that the condition (1) holds in U, and $\left(e_i^{\beta_i}\right)_{\beta_i=1}^{m_i}$ is a basis of V_i.

Definition (Bursztyn, Cueca and Mehta, 2025)

A **morphism of algebro-geometric $\mathbb{N}$-graded manifolds of degree n** $\Psi\colon \mathcal{M} \to \mathcal{N}$ is a morphism of ringed spaces, given by a pair $\Psi = (\psi, \psi^\sharp)$, where $\psi\colon M \to N$ is a smooth map and $\psi^\sharp\colon \mathscr{C}_{\mathcal{N}} \to \psi_* \mathscr{C}_{\mathcal{M}}$ is morphism of sheaves of algebras over N which preserves the degrees.

Algebro-geometric $\mathbb{N}$-graded manifolds of degree n with these morphisms form the category $\mathrm{Man}^n_{\mathrm{alg}}$.

- The parity reversing functor is the endofunctor in the category of $\mathbb{Z}_2$-graded vector spaces defined by

$$\Pi\colon V \to \Pi V\,, \quad (\Pi V)_0 = V_1\,, \quad (\Pi V)_1 = V_0\,.$$

- It relates the exterior and symmetric algebras of V as follows:

$$\mathrm{S}^\bullet V = \bigwedge\nolimits^\bullet \Pi V\,, \quad \bigwedge\nolimits^\bullet V = \mathrm{S}^\bullet \Pi V\,.$$

- Consequently, the local form (1) of the sheaf on an $\mathbb{N}$-manifold can be rewritten as

$$\mathscr{C}_{\mathcal{M}}|_U \simeq \mathscr{C}^\infty_U \otimes \bigwedge\nolimits^\bullet W\,, \quad W := \Pi V\,.$$

From the algebro-geometric approach to homogeneity

- Note that an algebro-geometric $\mathbb{N}$-graded manifold is naturally equipped with a weight vector field ∇ locally given by

$$\nabla = \sum_{j=1}^{n} \sum_{\beta_j=1}^{m_j} j\, e_j^{\beta_j} \frac{\partial}{\partial e_j^{\beta_j}} .$$

- The flow of ∇ defines an action of the monoid $(\mathbb{R}, \cdot)$ on $\mathcal{M}$ given by

$$h_t \left(x^a, e_i^{\beta_i} \right) = \left(x^a, t^i e_i^{\beta_i} \right) .$$

- In particular, h_0 can be identified with the projection onto the body $M = |\mathcal{M}|$.
- Additionally, note that h_{-1} acts as the parity operator.

Homogeneity structures

Definition (Jóźwikowski and Rotkiewicz, 2016)

A **homogeneity structure on a supermanifold** $\mathcal{M}$ is a smooth action $h\colon \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ of the multiplicative monoid $(\mathbb{R}, \cdot)$ of real numbers, i.e., h is a morphism of supermanifolds defined by a collection of maps $h_t\colon \mathcal{M} \to \mathcal{M}$, $t \in \mathbb{R}$ such that $h_{ts}^* = h_t^* \circ h_s^*$ for any $s, t \in \mathbb{R}$ and $h_1 = \mathrm{Id}_{\mathcal{M}}$.

Homogeneity structures

Definition (Jóźwikowski and Rotkiewicz, 2016)

A morphism of two homogeneity structures $(\mathcal{M}_1, h_1)$ and $(\mathcal{M}_2, h_2)$ is a morphism of supermanifolds $\Phi\colon \mathcal{M}_1 \to \mathcal{M}_2$ intertwining the actions h_1 and h_2. The category of homogeneity structures on supermanifolds has these morphisms and pairs $(\mathcal{M}, h)$ as objects.

Homogeneity structures

Definition (Jóźwikowski and Rotkiewicz, 2016)

A (local) function f on $\mathcal{M}$ is called **homogeneous of weight** $w \in \mathbb{N}$ if

$$h_t^*(f) = t^w \cdot f\,, \quad \forall\, t \in \mathbb{R}\,.$$

Proposition (Jóźwikowski and Rotkiewicz, 2016)

Given a homogeneity structure $h\colon \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ *on a supermanifold* $\mathcal{M}$*, one can always find an atlas with homogeneous coordinates on* $\mathcal{M}$*.*

Definition

A **homogeneous $\mathbb{N}$-graded supermanifold** is a supermanifold $\mathcal{M}$ equipped with a homogeneity structure $h\colon \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ such that $h_0\colon \mathcal{M} \to M$ is the projection onto the body of M and h_{-1} acts as the parity operator. By the degree n of $(\mathcal{M}, h)$ we will refer to the maximum weight among the set of weights of homogeneous coordinates.

Homogeneous $\mathbb{N}$-graded supermanifolds of degree n with morphims between their homogeneity structures form the category $\mathrm{HSMan}_{\mathbb{N}}^{n}$.

This is the definition of N-manifold used by Ševera and Roytenberg.

Using an atlas of homogeneous coordinates, *h* defines a grading on the sheaf of functions.

Theorem

The categories $\mathrm{Man}^n_{\mathrm{alg}}$ *and* $\mathrm{HSMan}^n_{\mathbb{N}}$ *are equivalent. Moreover, on each* $\mathbb{N}$*-graded manifold* $\mathcal{M}$*, the degrees of homogeneous functions defined by the homogeneity structure* h *and by the grading of the sheaf* $\mathscr{C}_{\mathcal{M}}$ *coincide.*

Definition (Bursztyn, Cueca and Mehta, 2025)

Let $\mathcal{M} = (M, \mathscr{C}_{\mathcal{M}})$ be an algebro-geometric $\mathbb{N}$-graded manifold of degree n and dimension $m_0 | \cdots | m_n$. Given an open subset $U \subseteq M$, a vector field of degree k on $\mathcal{M}|_U$ is a degree k derivation X of $\mathscr{C}_{\mathcal{M}}(U)$, i.e., an $\mathbb{R}$-linear map $X\colon \mathscr{C}_{\mathcal{M}}(U) \to \mathscr{C}_{\mathcal{M}}(U)$ with the property that, for all $f, g \in \mathscr{C}_{\mathcal{M}}(U)$ with f homogeneous, $\deg(Xf) = \deg(f) + k$ and

$$X(fg) = X(f)g + (-1)^{k \deg(f)} f X(g)\,.$$

The sheaf of all vector fields on $\mathcal{M}$ is denoted by $\mathscr{T}^{\bullet}_{\mathcal{M}}$. The graded commutator of vector fields is defined by

$$[X, Y] = XY - (-1)^{\deg(X)\deg(Y)} YX$$

for homogeneous vector fields X and Y, and extended to any vector fields by linearity.

Definition (Bursztyn, Cueca and Mehta, 2025)

A **distribution** on $\mathcal{M}$ of rank $d_0 | \cdots | d_n$ is a graded subsheaf $D \subseteq \mathscr{T}^\bullet_{\mathcal{M}}$ of $\mathscr{C}_{\mathcal{M}}$-modules such that any point in M admits an open neighborhood U in which $D(U)$ is generated by linearly independent vector fields

$$\{X_k^{i_k}\}\,, \quad 0 \leq k \leq n\,, \quad 1 \leq i_k \leq d_k\,.$$

which are homogeneous with $\deg(X_k^{i_k}) = -k$.

Proposition

There is a one-to-one correspondence between homogeneous distributions on a homogeneous $\mathbb{N}$-graded supermanifold and distributions on an algebro-geometric $\mathbb{N}$-graded manifold.

From the Frobenius theorem for manifolds with weight vector fields we can recover the main result by Bursztyn, Cueca and Mehta:

Corollary

Let D be an involutive distribution of rank $d_0 | \cdots | d_n$ on an algebro-geometric $\mathbb{N}$-graded manifold $\mathcal{M} = (M, \mathscr{C}_{\mathcal{M}})$ of degree n and dimension $m_0 | \cdots | m_n$. Then, around any point in M there is a chart $(U; x^\alpha, e_j^{\beta_j})$ such that

$$D(U) = \left\langle \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial e_j^{\beta_j}} \mid 1 \leq \alpha \leq d_0\,,\ 1 \leq j \leq n\,,\ 1 \leq \beta_j \leq d_j \right\rangle .$$

The proof by Bursztyn, Cueca and Mehta is much more intricate. They have several non-trivial theorems, lemmas and propositions showing that:

- There is an equivalence of categories associating to each $\mathbb{N}$-graded manifold $\mathcal{M} = (M, \mathscr{C}_{\mathcal{M}})$ an **admissible coalgebra bundle**, i.e
  - a graded vector bundle $E = \bigoplus_{i=-n}^{-1} E_i \to M$ with
  - a degree preserving VB map $\mu \colon E \to E \otimes E$ which is coassociative and graded cocommutative, namely,

$$(\mathrm{Id} \otimes \mu) \circ \mu = (\mu \otimes \mathrm{Id}) \circ \mu, \quad \mu = \tau \cdot \mu,$$

where $\tau \in S_2$ acts by $\tau\,(e \otimes e') = (-1)^{i+j} e' \otimes e,$ for $e \in E_i, e' \in E_j$.

The proof by Bursztyn, Cueca and Mehta is much more intricate. They have several non-trivial theorems, lemmas and propositions showing that:

- A distribution in $\mathcal{M}$ is equivalent to a collection of vector subbundles $C_k \subseteq \mathrm{Der}(E^*)_{-k} \subseteq \mathrm{Der}(E^*)$ satisfying certain properties.
- An involutive distribution is equivalent to an involutive subbundle $F \subseteq \mathrm{T}M$ and a collection of flat connections ∇^i on E^*_{-i} satisfying

  $$\nabla_X^{i+j}\left(\mu^*\left(e, e'\right)\right) = \mu^*\left(\nabla_X^i e, e'\right) + \mu^*\left(e, \nabla_X^j e'\right),$$

  for all $e \in \Gamma(E^*_{-i}),\ e' \in \Gamma(E^*_{-j}), X \in \Gamma(F)$.

Conclusions

Summary

- Weight vector fields and homogeneity structures allows extending results to the graded realm by means of standard differential-geometric tools.
- We have shown the existence of coordinates which are simultaneously homogeneous and canonical for differential forms.

Future research

- This formalism is specially useful for combining a grading with another compatible geometric structure, e.g. VB-groupoids or VB-algebroids.
- Multiple gradings: ∇_1 and ∇_2 such that $[\nabla_1, \nabla_2] = 0$
- Homogeneous Poisson structures and homogeneous Weinstein splitting coordinates
- Homogeneous multisymplectic forms

Main references

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Thank you for your attention!

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