Hybrid Routhian reduction for simple hybrid forced Lagrangian systems

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Introduction

- Dimensionality reduction for large scale systems is a hot topic within the automatic control and robotics communities.
- The construction of methods for reduction of dimensionality permits, for instance, fast computations for the generation of optimal trajectories in optimal control problems of mechanical systems.
- A key element in the reduction is a Lie group of symmetries.

Simple hybrid systems

- Hybrid systems are dynamical systems with continuous-time and discrete-time components on its dynamics.
- In our case, the continuous dynamics will be the one corresponding to a forced mechanical system.
- Simple hybrid systems have been mainly employed for the understanding of locomotion gaits in bipeds and insects.

A brief recall of geometric mechanics I

- Let Q be an n-dimensional differentiable manifold with local coordinates (q^i), which represents the space of positions.
- TQ is the space of positions and velocities, with coords. $(q^i, \dot{q}^i)$.
- Given a Lagrangian function $L : TQ \to \mathbb{R}$, the dynamics of a mechanical system can be determined by its the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0$$

- The Lagrangian L is said to be regular if $\det \mathcal{M} \neq 0$, where

$$\mathcal{M} := \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right).$$

A brief recall of geometric mechanics II

- In addition, the motion of the system may be influenced by an external force, geometrically represented by a 1-form $F = F_i(q, \dot{q}^i)\mathrm{d}q^i$ on TQ.
- The dynamics of the forced Lagrangian system (L, F) are given by the forced Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = F_i(q, \dot{q}),$$

or by the integral curves of

$$X_L^F(q^i, \dot{q}^i) = \left(q^i, \dot{q}^i; \dot{q}^i, \mathcal{M}^{-1}\left(F_i + \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j\right)\right).$$

Simple hybrid systems

- A **simple hybrid system** is characterized by a tuple $\mathscr{L} = (D, X, S, \Delta)$, where:
  - D is a smooth manifold, called the **domain**, where the dynamics evolve,
  - X is a smooth vector field on D, representing the dynamics,
  - S is an embedded submanifold of D with co-dimension 1, called the **switching surface**, which can be thought of as the wall where the impact takes place,
  - $\Delta : S \to D$ is a smooth embedding, called the **impact map**, which represents the dynamics at the instant of the impact.
- Its dynamics are given by:

$$\begin{cases} \dot{\gamma}(t) = X(\gamma(t)), & \gamma(t) \notin S, \\ \gamma^+(t) = \Delta(\gamma^-(t)) & \gamma^-(t) \in S, \end{cases}$$

where $\gamma^{\pm}(t) := \lim_{\tau \to t^{\pm}} x(\tau)$.

Simple hybrid forced Lagrangian systems

- A simple hybrid system is a **simple hybrid forced Lagrangian system** if it is of the form $\mathscr{L}_F := (TQ, X_L^F, S, \Delta)$, where X_L^F is the dynamical vector field of the forced Lagrangian system (L, F).
- Given a smooth constraint function $h : Q \to \mathbb{R}$ such that $h^{-1}(0)$ is a smooth submanifold, we can construct a domain, a switching surface and an impact map explicitly explicitly:
    - $D = \{(q, \dot{q}) \in TQ : h(q) \geq 0\}$
    - $S = \{(q, \dot{q}) \in TQ : h(q) = 0,\ dh_q \dot{q} \geq 0\}$, where $dh_q = \frac{\partial h}{\partial q}$.
    - $\Delta(q, \dot{q}) = (q, P(q, \dot{q}))$, where

$$P(q, \dot{q}) = \dot{q} - (1 + e) \frac{dh_q \dot{q}}{dh_q \mathcal{M}^{-1} dh_q^T} \mathcal{M}^{-1} dh_q^T,$$

and e is the elastic constant.

Tangent action and momentum map

- Consider a Lie group action $\psi : G \times Q \to Q$.
- The tangent lift action $\psi^{TQ} : G \times TQ \to TQ$ is given by $\psi_g^{TQ}(q, \dot{q}) = \left(\psi_g(q), \frac{\partial \psi_g}{\partial q}(q)\dot{q}\right)$.
- We will assume that ψ^{TQ} is an **hybrid action** on $\mathscr{L}_F$, i.e.,
  - $L \circ \psi^{TQ} = L$,
  - ψ^{TQ} restricts to an action $\psi_g^{TQ}\mid_S$ of G on S,
  - $\Delta \circ \psi_g^{TQ}\mid_S = \psi_g^{TQ} \circ \Delta$.
- Let $\mathfrak{g}$ denote the Lie algebra of G, and $\mathfrak{g}^*$ its dual.
- ψ^{TQ} has a natural **momentum map** $J_L : TQ \to \mathfrak{g}^*$, given by

$$J_L(q, \dot{q})(\xi) = \frac{\partial L}{\partial \dot{q}^i}\xi^i,$$

for each $\xi \in \mathfrak{g}$, where $\xi^i \frac{\partial}{\partial q^i}$ is the generator of the ξ-action on Q.

Reduction of a simple hybrid forced system I

- A case of special interest with regards to applications is when $Q = \mathbb{S}^1 \times M$, where M is called the shape space.
- This is often the situation when dealing with simple models of bipedal walkers.
- The action is now simply $(\theta, x) \mapsto (\theta + \alpha, x)$.
- Consider a forced Lagrangian system (L, F) on TQ with cyclic coordinate θ, i.e., $L = L(x)$, where (x^i) are the coordinates in M, and $F = F_i(x, \dot{x}, \dot{\theta})\mathrm{d}x^i$.
- The momentum map in this case is simply $J_F = \frac{\partial L}{\partial \dot{\theta}}$.
- Take a value μ of the momentum map. One can use the relation $\frac{\partial L}{\partial \dot{\theta}} = \mu$ to express θ as a function of $x, \dot{x}$ and μ.

Reduction of a simple hybrid forced system II

- We can reduce $\mathscr{L}_F = (TQ, X_L^F, S, \Delta)$ to $\mathscr{L}_F^\mu = (M_\mu, X_{R^\mu}^{F_\mu}, S_\mu, \Delta_\mu)$, where
  - The reduced velocity space is $M_\mu := J_L^{-1}(\mu)/\mathbb{S}^1$.
  - The Routhian is $R^\mu(x, \dot{x}) = \left[L(\dot{\theta}, x, \dot{x}) - \mu\dot{\theta}\right]\Big|_{\dot{\theta}=\dot{\theta}(x,\dot{x},\mu)}$.
  - The reduced external force is $F_\mu(x, \dot{x}) = F(x, \dot{x}, \dot{\theta})|_\mu$.
  - The reduced switching surface is $S_\mu = J_L|_S^{-1}(\mu)/\mathbb{S}^1$.
  - Δ induces a reduced impact map $\Delta_\mu : S_\mu \to M_\mu$.
- Any solution of $\mathscr{L}_F$ with momentum μ projects into a solution of $\mathscr{L}_F^\mu$.
- Conversely, any solution of $\mathscr{L}_F^\mu$ is the projection of a solution of $\mathscr{L}_F$ with momentum μ.

Example (Billiard with dissipation)

- Consider a particle of mass m in the plane which is free to move inside the surface defined by $x^2 + y^2 = 1$, i.e., $h(x, y) = x^2 + y^2 - 1$.
- The Lagrangian of the system is $L(x, y, \dot{x}, \dot{y}) = \dfrac{1}{2} m(\dot{x}^2 + \dot{y}^2)$, and the external force is $F = 2c(\dot{x}xy - \dot{y}x^2)\mathrm{d}x - 2c(\dot{y}xy - \dot{x}y^2)\mathrm{d}y$.
- In polar coordinates, $L(\theta, r, \dot{\theta}, \dot{r}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$ and $F(\theta, r, \dot{\theta}, \dot{r}) = -2cr^3\dot{\theta}\mathrm{d}r$.
- Clearly, $J_L(r, \dot{r}, \theta, \dot{\theta}) = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ is preserved.
- The Routhian and the reduced force take the form $R^\mu(r, \dot{r}) = \frac{m}{2}\dot{r}^2 - \frac{\mu^2}{2mr^2}$ and $F_\mu = -2cr\frac{\mu}{m}\mathrm{d}r$.
- Suppose that an elastic impact ($e = 1$) occurs at $S = \left\{x^2 + y^2 = 1,\ x\dot{x} + y\dot{y} \geq 0\right\}$.
- We conclude that $\dot{r}^+ = -\dot{r}^-$ and $\dot{\theta}^+ = \dot{\theta}^-$.

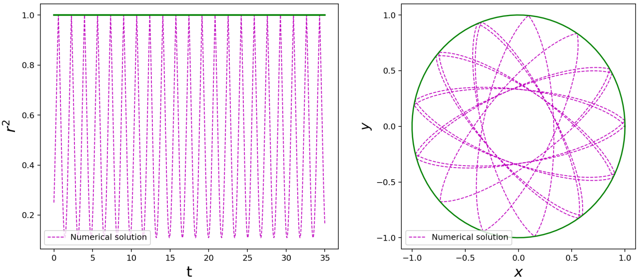

Figure: Simulation for $c = 2$. The first figure corresponds with the reduced trajectory while the second figure with the reconstructed solution.

We acknowledge Manuela Gamonal for her help with the numerical simulations.

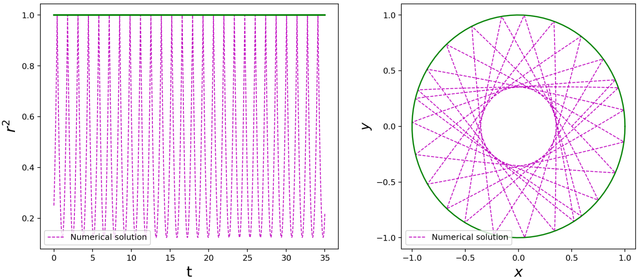

Figure: Simulation for $c = 0.20$. The first figure corresponds with the reduced trajectory while the second figure with the reconstructed solution.

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Thank you!

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