Nonsmooth Herglotz Principle

A variational principle for dissipative systems with impacts

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Motivation

- Several dynamical systems in physics and optimal control theory can be described by means of a variational principle.
- Having a variational principle permits to obtain:
  - numerical methods,
  - symmetries and conservation laws,
  - a geometric framework $\rightsquigarrow$ geometric methods,
- In systems with collisions trajectories are no longer smooth.
- We have developed a variational principle describing the dynamics of mechanical systems with dissipation and impacts.

Principle of least action

- Let Q denote the space of positions of a mechanical system, and TQ the space of positions and velocities.
- Let $L\colon TQ \to \mathbb{R}$ denote the Lagrangian function of the system.
- The action functional is given by

$$\mathcal{A}[q] = \int_{t_0}^{t_1} L(q(t), \dot{q}(t)) \, \mathrm{d}t \,,$$

for each smooth curve $q\colon [t_0, t_1] \subseteq \mathbb{R} \to Q$.

Principle

The trajectories of a dynamical system on Q with Lagrangian function $L\colon TQ \to \mathbb{R}$ are the curves such that

$$\delta\mathcal{A} = 0 \,.$$

Euler–Lagrange equations

Theorem

A smooth curve $q\colon [t_0, t_1] \subseteq \mathbb{R} \to Q$ *is an extremal of the action* $\mathcal{A}$ *iff it satisfies the Euler–Lagrange equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0\,.$$

Example (mechanical Lagrangian)

For a Lagrangian function of the form

$$L(q, \dot{q}) = \frac{m}{2}\dot{q}^2 - V(q)\,,$$

the Euler–Lagrange equations yield Newton's Second Law:

$$m\ddot{q} = F\,,$$

where $F = -\operatorname{grad} V$.

Action-dependent Lagrangians

- From the fundamental theorem of calculus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(q(t)) = L(q(t), \dot{q}(t))\,.$$

- Now suppose that the Lagrangian function depends on the action:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(q(t)) = L(q(t), \dot{q}(t), \mathcal{A}(q(t)))\,.$$

- This will lead to extra terms on the Euler–Lagrange equations, which allows to describe dissipative phenomena.

Action-dependent Lagrangians

Theorem

Given an action-dependent Lagrangian function $L\colon TQ \times \mathbb{R} \to \mathbb{R}$, a smooth curve $q\colon [t_0, t_1] \subseteq \mathbb{R} \to Q$ is an extremal of the action $\mathcal{A}$ iff it satisfies the Herglotz–Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} - \frac{\partial L}{\partial \dot{q}}\frac{\partial L}{\partial \mathcal{A}} = 0\,.$$

Energy dissipation

- The energy function is

$$E_L(q, \dot{q}, \mathcal{A}) = \dot{q}\frac{\partial L}{\partial \dot{q}} - L(q, \dot{q}, \mathcal{A})\,.$$

- Its evolution is given by

$$\frac{\mathrm{d}E_L}{\mathrm{d}t} = \frac{\partial L}{\partial \mathcal{A}} E_L\,,$$

- Hence,

$$E_L\big(q(t), \dot{q}(t), \mathcal{A}(t)\big) = E_L\big(q(0), \dot{q}(0), \mathcal{A}(0)\big)\, e^{\int_0^t \frac{\partial L}{\partial \mathcal{A}}\left(q(t), \dot{q}(t)\right)\,\mathrm{d}t}$$

Example (The damped harmonic oscillator)

Consider the action-dependent Lagrangian

$$L(q, \dot{q}, \mathcal{A}) = \frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2 - \gamma\mathcal{A}\,.$$

The Herglotz–Euler–Lagrange equations yield

$$m\ddot{q} + kq + \gamma\dot{q} = 0\,.$$

The energy

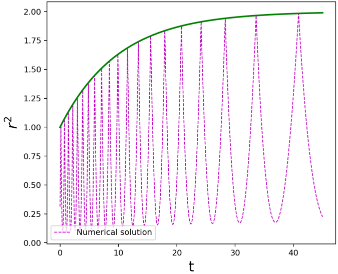
$$E_L = \frac{m}{2}\dot{q}^2 + \frac{k}{2}q^2 + \gamma\mathcal{A}$$

decreases exponentially, namely,

$$E_L\big(q(t), \dot{q}(t), \mathcal{A}(t)\big) = E_L\big(q(0), \dot{q}(0), \mathcal{A}(0)\big)\, e^{-\gamma t}\,.$$

Nonsmooth trajectories

- So far we have assumed that the trajectories considered were smooth (at least $\mathcal{C}^2$) curves.
- However, at points were the system experiences an impact it is not realistic to assume the curve to be smooth.
- Thus, instead we will have to consider curves which are $\mathcal{C}^0$ and piecewise $\mathcal{C}^2$.



Nonsmooth principle of least action

- Suppose that the admissible configurations are in a submanifold $C \subset Q$ and that the impacts take place at the boundary ∂C.
- For simplicity's sake suppose that the system experiences a single impact at time $t^* \in (t_0, t_1)$.
- Let $\mathcal{Q}$ denote the set of all curves $q\colon [t_0, t_1] \subseteq \mathbb{R} \to C$ which are $\mathcal{C}^0$ in (t_0, t_1) and $\mathcal{C}^2$ in $(t_0, t^*) \cup (t^*, t_1)$.

Nonsmooth principle of least action

Theorem (Fetecau, Marsden, Ortiz and West, 2003)

A curve $q \in \mathcal{Q}$ is an extremal of the action $\mathcal{A}$ iff

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t)) - \frac{\partial L}{\partial q}(q(t), \dot{q}(t)) = 0, \qquad \textit{for} \quad t \in [t_0, t^*) \cup (t^*, t_1],$$

$$\lim_{t \to t^{*-}} \frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t)) \cdot v = \lim_{t \to t^{*+}} \frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t)) \cdot v, \quad \forall\, v \textit{ tangent to } \partial C,$$

$$\lim_{t \to t^{*-}} E_L(q(t), \dot{q}(t)) = \lim_{t \to t^{*+}} E_L(q(t), \dot{q}(t)).$$

Nonsmooth Herglotz principle

Theorem (Colombo, de León and L.-G., 2022)

Given an action-dependent Lagrangian function $L\colon TQ\times\mathbb{R}\to\mathbb{R}$*, a curve* $q\in\mathcal{Q}$ *is an extremal of the action* $\mathcal{A}$ *iff*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q}(q(t),\dot q(t)) - \frac{\partial L}{\partial q}(q(t),\dot q(t)) - \frac{\partial L}{\partial \dot q}\frac{\partial L}{\partial \mathcal{A}}(q(t),\dot q(t)) = 0, \quad t\neq t^*\,,$$

$$\lim_{t\to t^{*-}}\frac{\partial L}{\partial \dot q}(q(t),\dot q(t))\cdot v = \lim_{t\to t^{*+}}\frac{\partial L}{\partial \dot q}(q(t),\dot q(t))\cdot v, \quad \forall\ v \text{ tangent to } \partial C\,,$$

$$\lim_{t\to t^{*-}} E_L(q(t),\dot q(t)) = \lim_{t\to t^{*+}} E_L(q(t),\dot q(t))\,.$$

Application: billiard with dissipation

- Let $Q = \mathbb{R}^2$ with cartesian coordinates (x, y).
- Consider a particle confined to the unit disk

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\right\},$$

with action-dependent Lagrangian function

$$L(x, y, \dot{x}, \dot{y}, \mathcal{A}) = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \gamma \mathcal{A}.$$

- Herglotz equations yield

$$\ddot{x} = -\gamma \dot{x}, \quad \ddot{y} = -\gamma \dot{y}.$$

Application: billiard with dissipation

- Integrating them we obtain

$$x(t) = x_0 + \frac{\dot{x}_0}{\gamma}\left(1 - e^{-t\gamma}\right),$$

$$y(t) = y_0 + \frac{\dot{y}_0}{\gamma}\left(1 - e^{-t\gamma}\right).$$

Application: billiard with dissipation

- The boundary of C is the unit circle

$$\partial C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\right\}.$$

- The velocities $(\dot{x}^+, \dot{y}^+)$ after the impact are given by

$$\dot{x}^+ = \frac{-\dot{x}^- x^2 + \dot{x}^- y^2 - 2\dot{y}^- xy}{x^2 + y^2},$$

$$\dot{y}^+ = \frac{-2\dot{x}^- xy + \dot{y}^- x^2 - \dot{y}^- y^2}{x^2 + y^2},$$

where $(\dot{x}^-, \dot{y}^-)$ are the velocities before the impact.

Application: billiard with dissipation

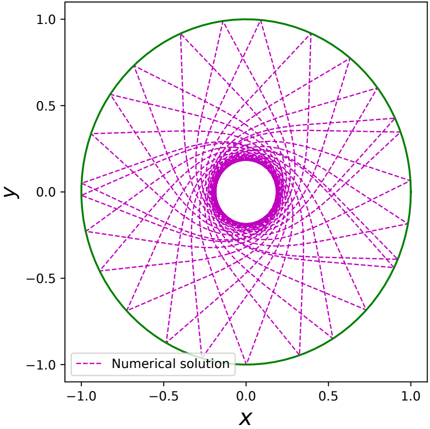


Figure: Numerical simulation for $\gamma = 10^{-4}$.

Conclusions and further work

- We have developed a non-differentiable Herglotz variational principle that allows us to describe mechanical systems with dissipation and impacts.
- Subsequent problems related to the results discussed in this work include:
  - The reduction of this type of systems when there are symmetries that leave the Lagrangian invariant.
  - The construction of variational integrators that preserve the qualitative behaviour of the system.
  - A geometric formulation for these systems $\rightsquigarrow$ hybrid contact Hamiltonian systems.

Main references

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Thanks for your attention!

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