Darboux theorem for homogeneous presymplectic and Pfaffian forms

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There are several situations in geometry and physics in which a $(\mathbb{N}, \mathbb{Z}, \mathbb{Z}_2, \mathbb{R}, \dots)$ grading appears:

* The algebra of differential forms with the wedge product.
* The spin of particles.
* Intensive/extensive variables in thermodynamics
* Symplectisation/Poissonisation of contact/Jacobi mfolds.
* Supermanifolds
* Higher tangent bundles

Theorem (Euler): Let $f: \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. The following statements are equivalent:

i) f is k-homogeneous ($k \in \mathbb{Z}$), namely

$$f(tx^1, \dots, tx^n) = t^k f(x^1, \dots, x^n) \qquad \forall t \in \mathbb{R} \setminus \{0\}.$$

ii) f is a solution of the PDE

$$k \cdot f = \sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i}.$$

In other words, homogeneous functions are eigenfunctions of

$$X = \sum_{i=1}^{n} x^i \, \partial_{x^i} . \qquad (*)$$

In particular, linear = 1-homogeneous

We can extend this idea to manifolds by considering a vector field X that is locally of the form $(*)$ in some coords.

Def.: A vector field ∇ on a manifold M is called a weight vector field if in a neighbourhood of every point of M there are local coordinates (x^a) such that

$$\nabla = \sum_{a=1}^{n} w_a \, x^a \partial_{x^a} ,$$

where $w_a =: \deg(x^a) \in \mathbb{R}$ is called the weight of x^a.
Such coordinates are called homogeneous coordinates.

The pair (M, ∇) is called a homogeneity manifold.

Def.: Let (M, ∇) be a homogeneity manifold.

A tensor field A on M is called w-homogeneous $(w \in \mathbb{R})$ if

$$\mathcal{L}_{\nabla} A = w \cdot A.$$

Examples of homogeneity manifolds

* A vector bundle $\pi : E \longrightarrow M$ and the Euler vector field ∇_E (the generator of homotheties on the fibers).

  In bundle coords., $\pi : (x^i, y^a) \longmapsto (x^i)$,

$$\nabla_E = \sum_a y^a \partial_{y^a} .$$

* The second-order tangent bundle

$$\tau : T^2M \ni (x^i, \dot{x}^i, \ddot{x}^i) \longmapsto (x^i) \in M$$

with $\deg(x^i) = 0$, $\deg(\dot{x}^i) = 1$, $\deg(\ddot{x}^i) = 2$.

* An exact symplectic manifold $(M, \omega = d\theta)$ with a Liouville vector field ∇, i.e.

$$\mathcal{L}_{\nabla} \omega = \omega .$$

* Weight vector fields with non-integer weights appear in BH thermodynamics

↳ F. Belgiorno, "Quasi-homogeneous thermodynamics and black holes", J. Math. Phys. 44, 1089 (2003)

Let (M, ∇) be a homogeneity mfold.

There are two different situations on an open subset $U \subseteq M$

* $\nabla|_U \neq 0$,

* $\exists\ x_0 \in U$ s.t. $\nabla(x_0) = 0$.

Remark: Any nowhere-vanishing vector field $X \in \mathfrak{X}(M)$ is a weight vector field. However, its weights are not canonical.

Indeed, since X is nowhere zero, $\exists$ local coords. (x^a) such that $X = \partial_{x^1}$. For any $\{w_1, \dots, w_n\} \subset \mathbb{R}$ with $w_1 \neq 0$, we can def. a new system of coords.

$$y^1 = e^{w_1 x^1}, \qquad y^i = e^{w_i x^1} x^i, \quad 2 \leq i \leq n$$

so that

$$X = \sum_{a=1}^{n} w_a y^a \partial_{y^a}, \quad \text{i.e.} \qquad \deg(y^a) = w_a.$$

On the other hand, in a neighbourhood of any point at which a weight vector field vanishes, its weights are canonical.

<u>Proposition (Grabowska & Grabowski, 2024):</u> $\nabla \in \mathfrak{X}(M)$ is a weight vector field on M iff $T_{x_0}X$ is diagonal $\forall x_0 \in M$ s.t. $\nabla(x_0) = 0$.

Let (x^a) be a system of homog. coords. around x_0, i.e.

$$\nabla = \sum_a w_a x^a \partial_{x^a}, \quad \text{with} \quad \Gamma := \{w_1, \dots, w_n\} \subset \mathbb{R}.$$

Then, any other system of homog. coords. around x_0 has weights in Γ.

Homogeneous Poincaré Lemma (Grabowska & Grabowski, '24):

Let (M, ∇) be a homogeneity mfold. Let $\omega \in \Omega^k(M)$ be a λ-homogeneous k-form $(k>0)$. Assume that $\nabla(x_0)=0$.

In a nbh. of x_0, $\exists$ $(k-1)$-form α s.t.

i) $d\alpha = \omega$,

ii) α is λ-homogeneous,

iii) $\alpha(x_0) = 0$.

Darboux theorem for homogeneous symplectic forms (G & G '24)

Let (M, ∇) be a homogeneity mfold., and let ω be a λ-homog. symplectic form on M. Then, around every $x_0 \in M$ s.t. $\nabla(x_0) = 0$, there is a system of homog. coords. (q^i, p_i) such that

$$\omega = \sum_i dp_i \wedge dq^i, \qquad \nabla = \sum_i \left(w_{q^i}\, q^i \partial_{q^i} + w_{p_i}\, p_i \partial_{p_i} \right).$$

Homogeneous straightening lemma (Grabowski & LG):

Let (M, ∇) be a homogeneity mfold., and let $X \in \mathcal{X}(M)$ be a $(-\lambda)$-homogeneous vector field. Assume that $\nabla(x_0) = 0$ and $X(x_0) \neq 0$ at $x_0 \in M$. Then, in a neighbourhood of x_0, there is a chart of homog. coords. $(U; z, y^i)$ such that

$$X = \partial_z, \qquad \nabla = \lambda z \partial_z + \sum_i w_i y^i \partial_{y^i}.$$

Def.: Let (M, ∇) be a homog. mfold. A (co)distribution $D \subset TM$ (resp. $D \subset T^*M$) is called **homogeneous** if the (co) tangent lift $d_T \nabla$ (resp. $d_{T^*} \nabla$) is tangent to D.

Theorem (Grabowski & Lg): $D \subset TM$ is homogeneous iff it is locally generated by homogeneous vector fields.

Corollary: $D^\circ \subset T^*M$ is homog. iff it is locally generated by homog. one-forms.

Homogeneous Frobenius theorem (Grabowski & LG):

Let (M, ∇) be a homog. mfold, and let D be an involutive distribution of rank k which is locally generated by homog. vector fields. Around every $x_0 \in M$ s.t. $\nabla(x_0) = 0$ $\exists$ homog. chart $(U; x^1, \dots, x^n)$ such that

$$D|_U = \langle \partial_{x^1}, \dots, \partial_{x^k} \rangle$$

and the slices

$$N = \{ x^{k+1} = \text{const.}, \dots, x^n = \text{const} \} \subset U$$

are integral submanifolds.

Def.: A presymplectic form ω on M is a closed 2-form of constant rank. Its characteristic distribution is given by

$$C_\omega = \ker \omega.$$

Proposition: C_ω is involutive. If (M, ∇) is a homog. mfold. and ω is homog., then C_ω is a homog. distrib.

Darboux theorem for homog. presymp. forms (Grabowski & LG):

Let (M, ∇) be a homogeneity mfold., and let ω be a λ-homog. presymp. form on M. Then, in a neighbourhood of each point $x_0 \in M$ s.t. $\nabla(x_0) = 0$, there exists a system of homog. coords. (q^i, p_i, z_a) s.t.

$$\omega = \sum_i dp_i \wedge dq^i ,$$

$$\nabla = \sum_i \left(w_{q^i} q^i \partial_{q^i} + w_{p_i} p_i \partial_{p_i} \right) + \sum_a w_{z_a} z^a \partial_{z^a} .$$

Def.: A one-form θ on a mfold. M^m is said to have odd class $2s+1 \leq m$ at $x \in M$ if

$$\theta \wedge (d\theta)^s(x) \neq 0 \quad \& \quad (d\theta)^{s+1}(x) = 0.$$

A contact form is a one-form of class $2s+1 = \dim M$ $\forall x \in M$.

Remark: If θ has constant class, $d\theta$ is presymplectic.

In the classical literature, one-forms are called Pfaffian forms.

Darboux thm. for homog. one-forms of odd class (Grabowski & LG):

Let (M, ∇) be a homogeneity mfold., and let θ be a λ-homog. 1-form of class $2s+1$. Then, in a neighbourhood of each point $x_0 \in M$ s.t. $\nabla(x_0) = 0$, there exists a system of homog. coords. (q^i, p_i, z, t_a) s.t.

$$\theta = dz + \sum_i p_i \, dq^i$$

Remark: coords. which are simultaneously homog. & Darboux may not exist in a neighbourhood $U \subseteq M$ s.t. $\nabla|_U \neq 0$.

Def.: Given a contact form η, the Reeb vector field is the unique $R \in \mathfrak{X}(M)$ s.t. $R \in \operatorname{Ker} d\eta$ & $\eta(R) = 1$.

Counterexample: $M = \mathbb{R}^3$, (x, y, z) canonical coords.

$$\eta = dz + y\,dx, \qquad \nabla = R = \partial_z$$

A homog. coord. ζ is a solution of the PDE $\nabla\zeta = W\zeta$

On the other hand, in Darboux coords. $(\tilde{q}^i, \tilde{p}_i, \zeta)$, $R(\zeta) = \nabla\zeta = 1$

<u>Def.:</u> A one-form β on a mfold. M^m is said to have <u>even class $2s+2$</u> $\leq m$ at x if

$$\beta \wedge (d\beta)^s(x) \neq 0 \quad \& \quad (d\beta)^{s+1}(x) \neq 0 \quad \& \quad \beta \wedge (d\beta)^{s+1}(x) = 0.$$

<u>Theorem (Darboux):</u> In a sufficiently small neighbourhood of x where ω has constant class, there are coords. (q^i, p_i, z^a) s.t.

$$\beta = \sum_{i=1}^{s+1} p_i \, dq^i$$

<u>Work in progress</u> If β is homog., are there coords. which are homog & Darboux simultaneously?

Future work

* Extending our results to supermanifolds.
* Homogeneous multisymplectic forms
* Applications to Pfaffian systems / exterior differential systems
  ↳ studying differential eqs. as ideals generated by differential forms
* Bi-homogeneity: ∇_1, ∇_2 s.t. $[\nabla_1, \nabla_2] = 0$.

References

K. Grabowska, J. Grabowski and Z. Ravanpak, "VB-structures and generalizations", Ann. Global Anal. Geom. 62, 1 (2022)

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Thank you for your attention!

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