Egorov-type semiclassical limits for open quantum systems with a bi-Lindblad structure

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GKSL generators

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the C^*-algebra of bounded operators on $\mathcal{H}$. We denote by $\mathcal{S}(\mathcal{H})$ the convex set of density matrices on $\mathcal{H}$, i.e. positive trace-one operators.

By *quantum dissipation* we refer to the irreversible, non-unitary component of the reduced dynamics of an open quantum system, described by a GKSL generator

$$\mathcal{L}(\rho) = -\frac{i}{\hbar}[H, \rho] + \sum_{k=1}^{N} \Big(L_k \rho L_k^\dagger - \tfrac{1}{2}\{L_k^\dagger L_k, \rho\} \Big),$$

where $H = H^\dagger$, $\{L_k\}_{k=1}^N \subset \mathcal{B}(\mathcal{H})$, and $\{A, B\} := AB + BA$ denotes the anticommutator.

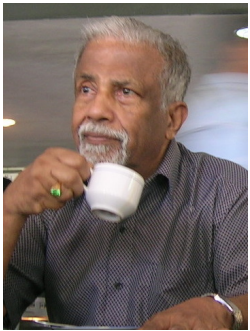
GKSL: Gorini–Kossakowski–Sudarshan–Lindblad



Vittorio Gorini
(1940–2025)



Andrzej Kossakowski
(1938–2021)



E. C. George Sudarshan
(1931–2018)



Göran Lindblad
(1940–2022)

Schrödinger and Heisenberg pictures

In the *Schrödinger picture*, the state evolves according to $\dot{\rho}(t) = \mathcal{L}(\rho(t))$; such $\mathcal{L}$ are precisely the generators of one-parameter semigroups $e^{t\mathcal{L}}$, $t \geq 0$, of completely positive and trace-preserving (CPTP) maps on $\mathcal{S}(\mathcal{H})$.

In the *Heisenberg picture*, observables evolve as $\dot{A}(t) = \mathcal{L}^\dagger(A(t))$, with[1]

$$\mathcal{L}^\dagger(A) = \frac{i}{\hbar}[H, A] + \sum_{k=1}^{N} \Big(L_k^\dagger A L_k - \tfrac{1}{2}\{L_k^\dagger L_k, A\} \Big).$$

$O \in \mathcal{B}(\mathcal{H})$ is a *(quantum) constant of motion* if $\mathrm{Tr}\big(\rho(t)O\big) = \mathrm{Tr}\big(\rho(0)O\big)$ for each solution of $\dot{\rho} = \mathcal{L}(\rho)$, i.e. $O \in \ker \mathcal{L}^\dagger$.

[1] $\mathrm{Tr}\big(\mathcal{L}(\rho)\, A\big) = \mathrm{Tr}\big(\rho\, \mathcal{L}^\dagger(A)\big)$ for all $\rho \in \mathcal{S}(\mathcal{H})$, $A \in \mathcal{B}(\mathcal{H})$.

Invariant commutative subalgebras

Theorem (Invariant commutative subalgebras for GKSL dynamics)

Let $\mathcal{L}$ be a Lindblad generator on $\mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} < \infty$, and let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a unital, commutative C^*-subalgebra. Suppose that

$$[H, A] = 0, \qquad [L_k, A] = [L_k^\dagger, A] = 0 \quad \text{for all } A \in \mathcal{A} \text{ and all } k.$$

Then $\mathcal{A} \subseteq \ker \mathcal{L}^\dagger$ and every observable $A \in \mathcal{A}$ is a quantum constant of motion.

Observables belonging to such a subalgebra have constant expectation values for all initial states and are insensitive to the dissipative dynamics. In this sense, they behave as robust, effectively classical quantities selected by the open-system evolution.

Example: dephasing Lindblad dynamics

$\mathcal{H} = \mathbb{C}^2$, with Pauli matrices $\sigma_x, \sigma_y, \sigma_z$; $H := \frac{\hbar\omega}{2}\sigma_z$, $L := \sqrt{\gamma}\,\sigma_z$, $\omega \in \mathbb{R}$, $\gamma > 0$:

$$\mathcal{L}^\dagger(A) = \frac{i\omega}{2}[\sigma_z, A] + \gamma(\sigma_z A \sigma_z - A),$$

$$\mathcal{L}^\dagger(\sigma_z) = 0, \qquad \mathcal{L}^\dagger(\sigma_x) = -2\gamma\sigma_x - \omega\sigma_y, \qquad \mathcal{L}^\dagger(\sigma_y) = -2\gamma\sigma_y + \omega\sigma_x.$$

This dynamics is known as *pure dephasing*: it suppresses the off-diagonal matrix elements of ρ in the eigenbasis of σ_z (i.e. quantum coherences) while leaving the diagonal entries (the populations) unchanged. The coherences decay exponentially with rate 2γ. Although dissipation is present, no energy exchange with the environment occurs.

$\mathcal{A} := \mathrm{span}\{\mathbf{1}, \sigma_z\}$ is invariant under $\mathcal{L}^\dagger$. Indeed, $\mathcal{L}^\dagger(\mathbf{1}) = 0$, $\mathcal{L}^\dagger(\sigma_z) = 0$.

The problem

Problem (sketch)

$(C, \alpha, \mathsf{h} = I_0, I_1, \ldots, I_n)$ is a completely integrable contact system, with classical dissipated quantities $I_0, \ldots, I_n$ coming from a bi-Hamiltonian Poisson–Lie system. Can one construct a Lindblad generator such that

1. *its semiclassical limit reproduces the contact Hamiltonian flow of (C, α, h);*
2. *its conserved observables $\widehat{I_0}, \ldots, \widehat{I_n}$ are quantizations of $I_0, \ldots, I_n$?*

Bi-Hamiltonian structures

Let M be a smooth manifold endowed with two Poisson tensors $\Pi_0, \Pi_1 \in \Gamma(\wedge^2 \mathsf{T}M)$, and $\Pi^\sharp(\alpha) \coloneqq \Pi(\alpha, \cdot)$.

Definition

The pair (Π_0, Π_1) is *compatible* if every linear combination $\Pi_\lambda \coloneqq \Pi_1 - \lambda\Pi_0$, $\lambda \in \mathbb{R}$, is a Poisson tensor (a *Poisson pencil*). A vector field X is *bi-Hamiltonian* if there exist $H_0, H_1 \in \mathscr{C}^\infty(M)$ such that $X = \Pi_0^\sharp(\mathrm{d}H_0) = \Pi_1^\sharp(\mathrm{d}H_1)$.

On any symplectic leaf where Π_0 is non-degenerate, one can introduce the recursion operator $N \coloneqq \Pi_1^\sharp \circ (\Pi_0^\sharp)^{-1}$; its eigenvalues are commuting functions in involution.

Example: Euler-top-type pencil

Identify $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$ with coordinates $m = (m_1, m_2, m_3)$. The Lie–Poisson structures of $\mathfrak{so}(3)$ and $\mathfrak{sl}(2,\mathbb{R})$:

$$\{m_1, m_2\}_0 = -m_3, \quad \{m_1, m_3\}_0 = m_2, \quad \{m_2, m_3\}_0 = -m_1,$$
$$\{m_1, m_2\}_1 = -m_2, \quad \{m_1, m_3\}_1 = m_3, \quad \{m_2, m_3\}_1 = -2m_1,$$

are compatible: $\pi_\lambda \coloneqq \pi_1 - \lambda\pi_0$ is a Poisson pencil. With the Casimir functions

$$C_0(m) = -\tfrac{1}{2}\big(m_1^2 + m_2^2 + m_3^2\big), \qquad C_1(m) = m_1^2 + m_2 m_3,$$

the vector field $X = \pi_0^\sharp \,\mathrm{d}C_1 = \pi_1^\sharp \,\mathrm{d}C_0$ is bi-Hamiltonian. Its trajectories are confined to the common level sets of (C_0, C_1).

Contact Hamiltonian systems

Contact geometry provides a natural framework for describing non-Hamiltonian classical dynamics, including certain dissipative systems.

Let (C, α) be a $(2n+1)$-dimensional co-oriented contact manifold, i.e. $\alpha \wedge (\mathrm{d}\alpha)^n$ is a volume form on C. The *Reeb vector field* R is uniquely determined by $\iota_R \mathrm{d}\alpha = 0$, $\alpha(R) = 1$. Given $\mathsf{h} \in \mathscr{C}^\infty(C)$, the *contact Hamiltonian vector field* X_h is defined by

$$\alpha(X_\mathsf{h}) = \mathsf{h}, \qquad \iota_{X_\mathsf{h}} \mathrm{d}\alpha = (\mathrm{d}\mathsf{h})(R)\,\alpha - \mathrm{d}\mathsf{h}.$$

The triple (C, α, h) is called a *contact Hamiltonian system*. The contact form α induces a Jacobi bracket

$$\{f, g\}_\alpha \coloneqq X_f(g) - g\,R(f), \qquad f, g \in \mathscr{C}^\infty(C).$$

Dissipated quantities

Definition

Let (C, α, h) be a contact Hamiltonian system. A function $f \in \mathscr{C}^\infty(C)$ is called a *dissipated quantity* if it is in Jacobi-involution with h, i.e. $\{f, \mathsf{h}\}_\alpha = 0$.

This notion is the natural replacement, in contact geometry, of conserved quantities in Hamiltonian dynamics.

Definition

A *completely integrable contact system* is a tuple $(C, \alpha, \mathsf{h} = f_1, f_2, \ldots, f_{n+1})$, where (C, α) is a $(2n+1)$-dimensional contact manifold and the $f_i \in \mathscr{C}^\infty(C)$ satisfy $\{f_i, f_j\}_\alpha = 0$ for all i, j, and $\mathrm{d}f_1, \ldots, \mathrm{d}f_{n+1}$ have rank at least n on a dense open subset of C. Each f_i is then a dissipated quantity.

Exact symplectic manifolds and homogeneity

An exact symplectic manifold (M, θ) carries the symplectic form $\omega \coloneqq -\mathrm{d}\theta$. Hamiltonian vector fields are defined by $\iota_{X_H}\omega = \mathrm{d}H$, and the *Liouville vector field* Δ is uniquely determined by $\iota_\Delta \omega = -\theta$.

Definition

A tensor field A on M is *homogeneous of degree* $w \in \mathbb{R}$ if $\mathcal{L}_\Delta A = w\, A$. A *homogeneous integrable system* is a tuple $(M, \theta, H = f_1, f_2, \ldots, f_n)$, where (M, θ) is a $2n$-dimensional exact symplectic manifold and $f_1, \ldots, f_n$ are homogeneous of degree 1, in involution, with $\mathrm{d}f_1, \ldots, \mathrm{d}f_n$ independent on a dense open subset. In particular, each f_i is a first integral of X_H.

The Hamiltonian flow of a 1-homogeneous function is invariant under the Liouville flow.

Symplectization

The symplectization of (C, α) is (C^Σ, θ), $C^\Sigma := C \times \mathbb{R}_{>0}$, $\theta := r\,\alpha$, $\Sigma(x, r) = x$, $\Delta = r\partial_r$, and $H^\Sigma(x, r) := r\,\mathsf{h}(x)$.

Proposition (Contact–homogeneous correspondence)

1. $f \mapsto f^\Sigma := r\,\Sigma^* f$ *is a bijection between* $\mathscr{C}^\infty(C)$ *and the* 1*-homogeneous functions on* C^Σ*, with* $X_{f^\Sigma} = X_f - rR(f)\partial_r$.
2. $\{f^\Sigma, g^\Sigma\}_\omega = \{f, g\}_\alpha^\Sigma$ *for all* $f, g \in \mathscr{C}^\infty(C)$.
3. f *is a dissipated quantity for* (C, α, h) *if and only if* f^Σ *is a first integral of* $(C^\Sigma, \theta, H^\Sigma)$.
4. $(C, \alpha, \mathsf{h} = f_1, \ldots, f_{n+1})$ *is a completely integrable contact system if and only if* $(C^\Sigma, \theta, H^\Sigma = f_1^\Sigma, \ldots, f_{n+1}^\Sigma)$ *is a homogeneous integrable system.*

Example: Euler-top-type pencil

Consider a symplectic manifold (S, ω) with $\omega = -\mathrm{d}\Theta$ exact, together with a Poisson map $\Phi\colon (S, \omega) \to (\mathfrak{so}(3)^*, \pi_0)$: take $S = \mathrm{SO}(3) \times \mathbb{R}^3_m$, where $\{\vartheta_1, \vartheta_2, \vartheta_3\}$ is the left-invariant Maurer–Cartan coframe of $\mathrm{SO}(3)$, with

$$\mathrm{d}\vartheta_1 = -\vartheta_2 \wedge \vartheta_3, \quad \mathrm{d}\vartheta_2 = -\vartheta_3 \wedge \vartheta_1, \quad \mathrm{d}\vartheta_3 = -\vartheta_1 \wedge \vartheta_2, \qquad \Theta := m_1\vartheta_1 + m_2\vartheta_2 + m_3\vartheta_3.$$

The projection $\Phi(g, m) = m$ is then a Poisson map onto $(\mathfrak{so}(3)^*, \pi_0)$. This realization is precisely $\mathrm{T}^*\mathrm{SO}(3)$ in its left trivialization.

Example: Euler-top-type pencil

The Liouville vector field is $\Delta = \sum_i m_i \partial_{m_i}$, and C_0, C_1 are homogeneous of degree 2. On $U = \{C_1 > 0\} \subset S$,

$$I_0^\Sigma = \sqrt{-2C_0} = \sqrt{m_1^2 + m_2^2 + m_3^2}, \qquad I_1^\Sigma = \sqrt{C_1} = \sqrt{m_1^2 + m_2 m_3}$$

are homogeneous of degree 1 and Poisson-commute. A third integral is supplied by the realization $\mathrm{T}^*\mathrm{SO}(3)$ itself: the *spatial* momentum $\mu := \mathrm{Ad}_g^*\, m$, with

$$\{m_i, \mu_j\}_\omega = 0, \qquad \{\mu_i, \mu_j\}_\omega = \varepsilon_{ijk}\, \mu_k.$$

With $I_2^\Sigma := \mu_3$, $(U, \Theta, H = I_0^\Sigma, I_1^\Sigma, I_2^\Sigma)$ is a homogeneous integrable system.

Liouville-transverse hypersurfaces

Lemma

Let (M, θ) be an exact symplectic manifold of dimension $2(n+1)$ and $C \subset M$ a hypersurface transverse to Δ. Then $\alpha := \theta|_C$ is a contact form on C.

Example (Euler-top-type pencil)

$C = \{m_1 = 1,\, m_2 m_3 > -1\} \subset U$ is transverse to Δ, and $\alpha := \Theta|_C = \vartheta_1 + m_2\vartheta_2 + m_3\vartheta_3$. The functions

$$\mathsf{h} = I_0 = \sqrt{1 + m_2^2 + m_3^2}, \qquad I_1 = \sqrt{1 + m_2 m_3}, \qquad I_2 = \mu_3|_C$$

pairwise commute with respect to $\{\cdot,\cdot\}_\alpha$, and $(C, \alpha, \mathsf{h} = I_0, I_1, I_2)$ is a completely integrable contact system.

Weyl quantization

Assume that M is an open subset of T^*Q, with canonical coordinates (x, ξ) and $\omega = \sum_{j=1}^{d} \mathrm{d}\xi_j \wedge \mathrm{d}x_j$.

Let $S^0(M)$ denote the set of symbols of order 0, i.e. smooth $a(x, \xi; \hbar)$ that are bounded together with all their derivatives, uniformly in $\hbar$.

The Weyl quantization $\mathrm{Op}_\hbar^{\mathrm{w}}$:

- maps $S^0(M)$ to bounded operators on $L^2(Q)$;
- maps Poisson brackets to commutators:

$$\frac{1}{i\hbar}\big[\,\mathrm{Op}_\hbar^{\mathrm{w}}(a), \mathrm{Op}_\hbar^{\mathrm{w}}(b)\big] = \mathrm{Op}_\hbar^{\mathrm{w}}\left(\{a, b\}_\omega\right) + \mathcal{O}(\hbar) \quad \text{in operator norm.}$$

Weyl quantization

The Weyl quantization $\mathrm{Op}_\hbar^{\mathrm{w}}$:

- on polynomials: $x_j \mapsto x_j$, $\xi_j \mapsto -i\hbar\,\partial_{x_j}$, symmetrically ordered;
- for a general $a \in S^0(M)$, with $Q = \mathbb{R}^d$:

$$\big(\mathrm{Op}_\hbar^{\mathrm{w}}(a)\psi\big)(x) \coloneqq \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{\hbar}(x-y)\cdot\xi}\, \psi(y)\, \mathrm{d}y\, \mathrm{d}\xi.$$

Semiclassical quantization

Let $\mathcal{A}_{\mathrm{cl}} \subset \mathscr{C}^\infty(C)$ denote the linear span of $I_0, \dots, I_n$.

Definition (Semiclassical quantization of the Jacobi commutative algebra)

A *semiclassical quantization* $(\mathcal{H}_\hbar, Q_\hbar)$ consists of a family of Hilbert spaces $\{\mathcal{H}_\hbar\}_{\hbar \in (0,\hbar_0]}$ and of linear maps $Q_\hbar \colon \mathcal{A}_{\mathrm{cl}} \to \mathcal{B}(\mathcal{H}_\hbar)$ such that, for all $f, g \in \mathcal{A}_{\mathrm{cl}}$: $Q_\hbar(\overline{f}) = Q_\hbar(f)^\dagger$, $Q_\hbar(1) = \mathbf{1}_{\mathcal{H}_\hbar}$, and

$$\lim_{\hbar \to 0} \left\| \frac{1}{i\hbar} \big[Q_\hbar(f), Q_\hbar(g) \big] - Q_\hbar\big(\{f, g\}_\alpha \big) \right\| = 0.$$

Here $Q_\hbar$ is obtained by first taking the homogeneous lift $f \mapsto f^\Sigma$ and then applying Weyl quantization: $Q_\hbar(f) = \mathrm{Op}^{\mathrm{w}}_\hbar(f^\Sigma)$.

Contact-compatible Lindblad generators

Denote $\widehat{f_\hbar} \coloneqq Q_\hbar(f)$ and $\widehat{I}_{j,\hbar} \coloneqq Q_\hbar(I_j)$.

Definition (Contact-compatible Lindblad generator)

A family $\{\mathcal{L}_\hbar\}_{\hbar\in(0,\hbar_0]}$ of GKSL generators on $\mathcal{B}(\mathcal{H}_\hbar)$, with Hamiltonians $\widehat{H}_\hbar$ and Lindblad operators $L_{k,\hbar}$, $k = 1, \dots, N(\hbar)$, is *contact-compatible* if:

(CC1) $\mathcal{L}_\hbar^\dagger\big(\widehat{I}_{j,\hbar}\big) = 0$ for each $j = 0, \dots, n$ (quantum constants of motion);

(CC2) for every $f \in \mathcal{A}_{\mathrm{cl}}$, (semiclassical contact limit)

$$\lim_{\hbar\to 0} \left\| \mathcal{L}_\hbar^\dagger\big(\widehat{f_\hbar}\big) - Q_\hbar\big(\{f, \mathsf{h}\}_\alpha\big) \right\| = 0.$$

Pseudodifferential Egorov criterion

Theorem (Pseudodifferential Egorov criterion for (CC2))

Let $(C, \alpha, \mathsf{h} = I_0, I_1, \dots, I_n)$ be a completely integrable contact system, and let $(M, \theta, H = I_0^\Sigma, I_1^\Sigma, \dots, I_n^\Sigma)$ be its symplectization. Suppose that:

1. *M is an open subset of T^*Q and $Q_\hbar$ is obtained by restricting $\mathrm{Op}_\hbar^{\mathrm{w}} \colon S^0(M) \to \mathcal{B}(\mathcal{H}_\hbar)$ to $\mathcal{A}_{\mathrm{cl}}$ via $f \mapsto f^\Sigma$;*
2. *$\widehat{H}_\hbar = \mathrm{Op}_\hbar^{\mathrm{w}}(H) + R_\hbar$, with $\|R_\hbar\| = \mathcal{O}(\hbar^2)$;*
3. *$L_{k,\hbar} = \mathrm{Op}_\hbar^{\mathrm{w}}(\ell_k) + \mathcal{O}(\hbar)$, with $\ell_k \in S^0(M)$ such that, for each k and every $f \in \mathcal{A}_{\mathrm{cl}}$, $\{\ell_k, f^\Sigma\}_\omega = 0$ and $\{|\ell_k|^2, f^\Sigma\}_\omega = 0$ on M, and $N(\hbar) = \mathcal{O}(1)$.*

Then the family $\{\mathcal{L}_\hbar\}$ satisfies (CC2) with respect to h, i.e. for every $f \in \mathcal{A}_{\mathrm{cl}}$,

$$\lim_{\hbar \to 0} \|\mathcal{L}_\hbar^\dagger(\widehat{f}_\hbar) - Q_\hbar(\{f, \mathsf{h}\}_\alpha)\| = 0.$$

Exact quantum constants of motion

Proposition

Under the same assumptions, if in addition the Lindblad data are chosen so that

$$[\widehat{H}_\hbar, \widehat{I}_{j,\hbar}] = 0, \qquad [L_{k,\hbar}, \widehat{I}_{j,\hbar}] = [L_{k,\hbar}^\dagger, \widehat{I}_{j,\hbar}] = 0 \quad \text{for all } j, k,$$

then (CC1) holds as well, i.e. $\mathcal{L}_\hbar^\dagger(\widehat{I}_{j,\hbar}) = 0$ *for* $j = 0, \dots, n$.

Quantization of Jacobi-commutative algebras

Let $\mathrm{Jacobi\text{-}alg}\langle S\rangle$ denote the smallest linear subspace of $\mathscr{C}^\infty(C)$ which contains S and is closed under pointwise multiplication and the Jacobi bracket.

Proposition

Let $\mathcal{A}_{\mathrm{cl}} = \mathrm{Jacobi\text{-}alg}\langle I_1, \ldots, I_n\rangle$ with $\{I_i, I_j\}_\alpha = 0$, $\{\mathsf{h}, I_i\}_\alpha = 0$, let $Q_\hbar$ be a semiclassical quantization of $\mathcal{A}_{\mathrm{cl}}$, and let $\{\mathcal{L}_\hbar\}$ be such that (1) $\mathcal{L}_\hbar^\dagger(\widehat{f}_\hbar) = Q_\hbar(\{f, \mathsf{h}\}_\alpha) + o(1)$ for all $f \in \mathcal{A}_{\mathrm{cl}}$, and (2) $[\widehat{H}_\hbar, \widehat{I}_{i,\hbar}] = 0$, $[L_{k,\hbar}, \widehat{I}_{i,\hbar}] = [L_{k,\hbar}^\dagger, \widehat{I}_{i,\hbar}] = 0$. Then the $\widehat{I}_{i,\hbar}$ generate a unital, commutative, invariant C^*-subalgebra $\mathcal{A}_\hbar = C^*(\widehat{I}_{1,\hbar}, \ldots, \widehat{I}_{n,\hbar})$, and every $A \in \mathcal{A}_\hbar$ is a quantum constant of motion.

Bi-Lindblad structures

Definition (Bi-Lindblad structure)

A *bi-Lindblad structure* associated with $(C, \alpha, \mathsf{h} = I_0, I_1, \dots, I_n)$ and $(\mathcal{H}_\hbar, Q_\hbar)$ consists of a pair of families of Lindblad generators $\{\mathcal{L}_\hbar^{(0)}\}$ and $\{\mathcal{L}_\hbar^{(1)}\}$ such that

(BL1) for every $\lambda \in [0,1]$, $\mathcal{L}_\hbar^{(\lambda)} := (1-\lambda)\,\mathcal{L}_\hbar^{(0)} + \lambda\,\mathcal{L}_\hbar^{(1)}$ is again a GKSL generator;
(BL2) $\mathcal{L}_\hbar^{(0)\,\dagger}(\widehat{I}_{j,\hbar}) = \mathcal{L}_\hbar^{(1)\,\dagger}(\widehat{I}_{j,\hbar}) = 0$ for each $j = 0, \dots, n$;
(BL3) for every $f \in \mathcal{A}_{\mathrm{cl}}$ and $a \in \{0,1\}$,

$$\lim_{\hbar \to 0} \left\| \mathcal{L}_\hbar^{(a)\,\dagger}(\widehat{f}_\hbar) - Q_\hbar\big(\{f, H_a^{\mathrm{eff}}\}_\alpha\big) \right\| = 0,$$

where $H_0^{\mathrm{eff}}, H_1^{\mathrm{eff}} \in \mathcal{A}_{\mathrm{cl}}$ are effective contact Hamiltonians whose classical flows on C are obtained from the bi-Hamiltonian Poisson–Lie data via the homogeneous lift and restriction.

Convex combinations

Proposition

Let $\{\mathcal{L}_\hbar^{(0)}\}$ *and* $\{\mathcal{L}_\hbar^{(1)}\}$ *define a bi-Lindblad structure, and fix* $\lambda \in [0,1]$*. Then, for each* $\hbar$*, the generator* $\mathcal{L}_\hbar^{(\lambda)} = (1-\lambda)\,\mathcal{L}_\hbar^{(0)} + \lambda\,\mathcal{L}_\hbar^{(1)}$ *is contact-compatible with* $(C, \alpha, \mathsf{h} = I_0, I_1, \ldots, I_n)$*, with respect to the effective Hamiltonian*

$$H_\lambda^{\mathrm{eff}} := (1-\lambda)\,H_0^{\mathrm{eff}} + \lambda\,H_1^{\mathrm{eff}} \in \mathcal{A}_{\mathrm{cl}}.$$

Example: Euler-top-type pencil

Here $\mathcal{A}_{\mathrm{cl}} := \mathrm{span}\{I_0, I_1, I_2\}$, and $Q_\hbar(f) := \mathrm{Op}^{\mathrm{w}}_\hbar(f^\Sigma)$ gives $\widehat{I}_{j,\hbar} = \mathrm{Op}^{\mathrm{w}}_\hbar(I_j^\Sigma)$ and $\widehat{\mathsf{h}}_\hbar := \widehat{I}_{0,\hbar}$. By the Dirac condition,

$$[\widehat{I}_{i,\hbar}, \widehat{I}_{j,\hbar}] = \mathcal{O}(\hbar^2), \qquad \mathcal{A}_\hbar := C^*(\widehat{I}_{0,\hbar}, \widehat{I}_{1,\hbar}, \widehat{I}_{2,\hbar}).$$

The quantization can be chosen so that $\widehat{I}_{0,\hbar}$, $\widehat{I}_{1,\hbar}$ and $\widehat{I}_{2,\hbar}$ commute exactly (sketch on the next slide). Then the joint spectral theorem yields

$$\mathcal{H}_\hbar = \bigoplus_\nu \mathcal{H}_{\hbar,\nu}, \qquad \widehat{I}_{j,\hbar}\big|_{\mathcal{H}_{\hbar,\nu}} = \lambda_{j,\nu}\,\mathbf{1}, \quad j \in \{0,1,2\}.$$

Semiclassically, the joint spectral parameter ν labels, up to Bohr–Sommerfeld conditions, the common level sets $\{I_0 = \lambda_{0,\nu},\ I_1 = \lambda_{1,\nu},\ I_2 = \lambda_{2,\nu}\}$ of the contact integrable system.

Example: Euler-top-type pencil

Let Y_i and Z_i be the left- and right-invariant vector fields on $\mathrm{SO}(3)$, and set $\widehat{m}_i \coloneqq -i\hbar\, Y_i$, $\widehat{\mu}_i \coloneqq -i\hbar\, Z_i$ on $L^2(\mathrm{SO}(3))$.

- Left- and right-invariant vector fields commute: $[\widehat{m}_i, \widehat{\mu}_j] = 0$.
- $\sum_i \widehat{m}_i^2 = \sum_i \widehat{\mu}_i^2 = -\hbar^2\Delta$ commutes with every $\widehat{m}_i$ and $\widehat{\mu}_j$.
- Hence $-\hbar^2\Delta$, $\widehat{C}_1 \coloneqq \widehat{m}_1^2 + \frac{1}{2}(\widehat{m}_2\widehat{m}_3 + \widehat{m}_3\widehat{m}_2)$ and $\widehat{\mu}_3$ commute exactly.

Take $\widehat{I}_{0,\hbar} \coloneqq \sqrt{-\hbar^2\Delta}$, $\widehat{I}_{1,\hbar} \coloneqq \sqrt{\widehat{C}_1}$ on the spectral subspace $\{\widehat{C}_1 > 0\}$, and $\widehat{I}_{2,\hbar} \coloneqq \widehat{\mu}_3$. The principal symbols of $\widehat{I}_{0,\hbar}$, $\widehat{I}_{1,\hbar}$, $\widehat{I}_{2,\hbar}$ are I_0^Σ, I_1^Σ, I_2^Σ.

Example: Euler-top-type pencil

With rates $\gamma_k > 0$ and bounded smooth functions $\varphi_k \colon \mathbb{R}^3 \to \mathbb{R}$,

$$L_{k,\hbar} \coloneqq \sqrt{\gamma_k}\, \varphi_k\big(\widehat{I}_{0,\hbar}, \widehat{I}_{1,\hbar}, \widehat{I}_{2,\hbar}\big) \in \mathcal{A}_\hbar, \qquad k = 1, \dots, N.$$

If $\widehat{H}_\hbar = \widehat{\mathsf{h}}_\hbar + \mathcal{O}(\hbar^2)$ commutes exactly with $\widehat{I}_{0,\hbar}, \widehat{I}_{1,\hbar}, \widehat{I}_{2,\hbar}$, then $\mathcal{L}_\hbar^\dagger(\widehat{I}_{j,\hbar}) = 0$, $j = 0, 1, 2$. With P_ν the projector onto $\mathcal{H}_{\hbar,\nu}$ and $\rho_{\nu\mu} \coloneqq P_\nu \rho P_\mu$, $\nu \neq \mu$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\nu\mu}(t)\Big|_{\mathrm{diss}} = -\frac{1}{2}\sum_{k=1}^{N} \big(\ell_{k,\nu} - \ell_{k,\mu}\big)^2\, \rho_{\nu\mu}(t), \qquad \ell_{k,\nu} \coloneqq \sqrt{\gamma_k}\, \varphi_k(\lambda_{0,\nu}, \lambda_{1,\nu}, \lambda_{2,\nu}).$$

The populations $\mathrm{Tr}(\rho P_\nu)$ are preserved. The open-system dynamics destroys coherences between states with different values of (I_0, I_1, I_2) without inducing transitions between the corresponding sectors: pure dephasing relative to $\mathcal{A}_\hbar$.

Example: Euler-top-type pencil

Take $H_0^{\mathrm{eff}} = I_0$ and $H_1^{\mathrm{eff}} = I_1$, with quantizations $\widehat{H}_{a,\hbar}$, and the *same* $L_{k,\hbar}$:

$$\mathcal{L}_\hbar^{(a)}(\rho) = -\frac{i}{\hbar}[\widehat{H}_{a,\hbar}, \rho] + \sum_{k=1}^{N} \Big(L_{k,\hbar}\rho L_{k,\hbar}^\dagger - \tfrac{1}{2}\{L_{k,\hbar}^\dagger L_{k,\hbar}, \rho\} \Big), \qquad a \in \{0,1\}.$$

- **(BL1)** every convex combination of $\mathcal{L}_\hbar^{(0)}$ and $\mathcal{L}_\hbar^{(1)}$ is a GKSL generator, since GKSL generators form a convex cone;
- **(BL2)** $\mathcal{L}_\hbar^{(a)\,\dagger}(\widehat{I}_{j,\hbar}) = 0$, since $\widehat{H}_{a,\hbar} = Q_\hbar(H_a^{\mathrm{eff}}) = \widehat{I}_{a,\hbar}$ and $L_{k,\hbar} \in \mathcal{A}_\hbar$ commute with the $\widehat{I}_{j,\hbar}$ (slide 24);

Example: Euler-top-type pencil

- **(BL3)** $\mathcal{L}_\hbar^{(a)\,\dagger}(\widehat{f_\hbar}) \to Q_\hbar(\{f, H_a^{\mathrm{eff}}\}_\alpha)$ for $f \in \mathcal{A}_{\mathrm{cl}}$, by the Egorov criterion, since the principal symbols ℓ_k depend only on $(I_0^\Sigma, I_1^\Sigma, I_2^\Sigma)$, so $\{\ell_k, f^\Sigma\}_\omega = \{|\ell_k|^2, f^\Sigma\}_\omega = 0$.

The Egorov limit of $\mathcal{L}_\hbar^{(1)}$ reproduces the contact flow of I_1, which projects under Φ onto the Euler-top vector field X, up to sign and the factor $1/(2\sqrt{C_1})$. $\mathcal{L}_\hbar^{(0)}$ quantizes the Casimir lift $\mathsf{h} = I_0$ and projects trivially.

Pseudodifferential Egorov criterion

Theorem (Pseudodifferential Egorov criterion for (CC2))

Let $(C, \alpha, \mathsf{h} = I_0, I_1, \dots, I_n)$ be a completely integrable contact system, and let $(M, \theta, H = I_0^\Sigma, I_1^\Sigma, \dots, I_n^\Sigma)$ be its symplectization. Suppose that:

1. *M is an open subset of T^*Q and $Q_\hbar$ is obtained by restricting $\mathrm{Op}_\hbar^{\mathrm{w}} \colon S^0(M) \to \mathcal{B}(\mathcal{H}_\hbar)$ to $\mathcal{A}_{\mathrm{cl}}$ via $f \mapsto f^\Sigma$;*
2. *$\widehat{H}_\hbar = \mathrm{Op}_\hbar^{\mathrm{w}}(H) + R_\hbar$, with $\|R_\hbar\| = \mathcal{O}(\hbar^2)$;*
3. *$L_{k,\hbar} = \mathrm{Op}_\hbar^{\mathrm{w}}(\ell_k) + \mathcal{O}(\hbar)$, with $\ell_k \in S^0(M)$ such that, for each k and every $f \in \mathcal{A}_{\mathrm{cl}}$, $\{\ell_k, f^\Sigma\}_\omega = 0$ and $\{|\ell_k|^2, f^\Sigma\}_\omega = 0$ on M, and $N(\hbar) = \mathcal{O}(1)$.*

Then the family $\{\mathcal{L}_\hbar\}$ satisfies (CC2) with respect to h, i.e. for every $f \in \mathcal{A}_{\mathrm{cl}}$,

$$\lim_{\hbar \to 0} \|\mathcal{L}_\hbar^\dagger(\widehat{f}_\hbar) - Q_\hbar(\{f, \mathsf{h}\}_\alpha)\| = 0.$$

Structure of the proof

Fix $f \in \mathcal{A}_{\mathrm{cl}}$ and write $\widehat{f}_\hbar = \mathrm{Op}^{\mathrm{w}}_\hbar(f^\Sigma)$. Let $\mathcal{D}_\hbar$ denote the dissipator in the Heisenberg picture:

$$\mathcal{L}^\dagger_\hbar(\widehat{f}_\hbar) = \frac{i}{\hbar}\big[\widehat{H}_\hbar, \widehat{f}_\hbar\big] + \mathcal{D}_\hbar(\widehat{f}_\hbar).$$

We first prove (CC2) (Steps 1–4), and then discuss (CC1) (Step 5).

Step 1 Dirac condition for $Q_\hbar$
Step 2 Hamiltonian part: $\frac{i}{\hbar}\big[\widehat{H}_\hbar, \widehat{f}_\hbar\big] = Q_\hbar\big(\{f, \mathsf{h}\}_\alpha\big) + \mathcal{O}(\hbar)$
Step 3 Estimate of the dissipative part: $\mathcal{D}_\hbar(\widehat{f}_\hbar) = \mathcal{O}(\hbar)$
Step 4 Conclusion of (CC2)
Step 5 Exact quantum constants of motion

Step 1: Dirac condition

By hypothesis, M is an open subset of T^*Q, and $Q_\hbar$ is obtained by composing the homogeneous lift $f \mapsto f^\Sigma$ with a semiclassical Weyl quantization $\mathrm{Op}^{\mathrm{w}}_\hbar \colon S^0(M) \to \mathcal{B}(\mathcal{H}_\hbar)$. The standard semiclassical symbolic calculus gives, for $a, b \in S^0(M)$,

$$\frac{1}{i\hbar}\big[\,\mathrm{Op}^{\mathrm{w}}_\hbar(a), \mathrm{Op}^{\mathrm{w}}_\hbar(b)\big] = \mathrm{Op}^{\mathrm{w}}_\hbar\left(\{a,b\}_\omega\right) + \mathcal{O}(\hbar)$$

in operator norm. Applying this with $a = f^\Sigma$, $b = g^\Sigma$ and using the Poisson–Jacobi correspondence $\{f^\Sigma, g^\Sigma\}_\omega = (\{f,g\}_\alpha)^\Sigma$, we obtain the Dirac condition (with an explicit $\mathcal{O}(\hbar)$ remainder):

$$\frac{1}{i\hbar}\big[\widehat{f}_\hbar, \widehat{g}_\hbar\big] = Q_\hbar\big(\{f,g\}_\alpha\big) + \mathcal{O}(\hbar).$$

Step 2: Hamiltonian part

By hypothesis, $\widehat{H}_\hbar = \mathrm{Op}^{\mathrm{w}}_\hbar(H) + R_\hbar$ with $\|R_\hbar\| = \mathcal{O}(\hbar^2)$. Then

$$\frac{i}{\hbar}\big[\widehat{H}_\hbar, \widehat{f}_\hbar\big] = \frac{i}{\hbar}\big[\,\mathrm{Op}^{\mathrm{w}}_\hbar(H), \mathrm{Op}^{\mathrm{w}}_\hbar(f^\Sigma)\big] + \frac{i}{\hbar}\big[R_\hbar, \mathrm{Op}^{\mathrm{w}}_\hbar(f^\Sigma)\big].$$

The second term is $\mathcal{O}(\hbar)$ in operator norm since $\|\,\mathrm{Op}^{\mathrm{w}}_\hbar(f^\Sigma)\|$ is uniformly bounded for $f^\Sigma \in S^0$. For the first term, the commutator expansion yields

$$\frac{i}{\hbar}\big[\,\mathrm{Op}^{\mathrm{w}}_\hbar(H), \mathrm{Op}^{\mathrm{w}}_\hbar(f^\Sigma)\big] = \mathrm{Op}^{\mathrm{w}}_\hbar\left(\{f^\Sigma, H\}_\omega\right) + \mathcal{O}(\hbar).$$

Using $\{f^\Sigma, H\}_\omega = (\{f, \mathsf{h}\}_\alpha)^\Sigma$, we obtain

$$\frac{i}{\hbar}\big[\widehat{H}_\hbar, \widehat{f}_\hbar\big] = Q_\hbar\big(\{f, \mathsf{h}\}_\alpha\big) + \mathcal{O}(\hbar) \quad \text{in operator norm.}$$

Step 3: dissipator at order $\hbar^0$

For each k, let

$$\mathcal{D}_{k,\hbar}(A) := L_{k,\hbar}^{\dagger} A L_{k,\hbar} - \tfrac{1}{2}\{L_{k,\hbar}^{\dagger} L_{k,\hbar}, A\},$$

so that $\mathcal{D}_\hbar = \sum_{k=1}^{N(\hbar)} \mathcal{D}_{k,\hbar}$. By the Weyl calculus, for each fixed k the summand $\mathcal{D}_{k,\hbar}(\widehat{f_\hbar})$ is a Weyl operator $\mathrm{Op}_\hbar^{\mathrm{w}}(d_{k,\hbar})$ up to $\mathcal{O}(\hbar)$ in operator norm (collecting the remainders coming from $L_{k,\hbar} - \mathrm{Op}_\hbar^{\mathrm{w}}(\ell_k) = \mathcal{O}(\hbar)$), where the symbol admits an expansion

$$d_{k,\hbar} = d_k^{(0)} + \hbar d_k^{(1)} + \mathcal{O}(\hbar^2).$$

The order-$\hbar^0$ term cancels identically,

$$d_k^{(0)} = \overline{\ell_k} f^{\Sigma} \ell_k - \tfrac{1}{2}\big(|\ell_k|^2 f^{\Sigma} + f^{\Sigma}|\ell_k|^2\big) = 0.$$

Step 3: dissipator at order $\hbar$

$d_k^{(1)}$ is a linear combination of Poisson brackets of the form

$$\{\ell_k, f^\Sigma\}_\omega \qquad \text{and} \qquad \{|\ell_k|^2, f^\Sigma\}_\omega.$$

The assumption that both brackets vanish therefore implies $d_k^{(1)} \equiv 0$ on M, and hence $\mathcal{D}_{k,\hbar}(\widehat{f}_\hbar) = \mathcal{O}(\hbar)$ in operator norm. Summing over k we obtain

$$\mathcal{D}_\hbar(\widehat{f}_\hbar) = \sum_{k=1}^{N(\hbar)} \mathcal{D}_{k,\hbar}(\widehat{f}_\hbar) = \mathcal{O}\big(N(\hbar)\hbar\big),$$

and since $N(\hbar) = \mathcal{O}(1)$ we conclude that $\|\mathcal{D}_\hbar(\widehat{f}_\hbar)\| = \mathcal{O}(\hbar)$ as $\hbar \to 0$.

Steps 4 and 5: conclusion

Step 4: Conclusion of (CC2). Combining Steps 2 and 3 we obtain

$$\mathcal{L}_\hbar^\dagger(\widehat{f}_\hbar) = \frac{i}{\hbar}[\widehat{H}_\hbar, \widehat{f}_\hbar] + \mathcal{D}_\hbar(\widehat{f}_\hbar) = Q_\hbar\big(\{f, \mathsf{h}\}_\alpha\big) + \mathcal{O}(\hbar),$$

hence $\lim_{\hbar\to 0} \|\mathcal{L}_\hbar^\dagger(\widehat{f}_\hbar) - Q_\hbar(\{f, \mathsf{h}\}_\alpha)\| = 0$, which is (CC2).

Step 5: Exact quantum constants of motion. Finally, (CC1), $\mathcal{L}_\hbar^\dagger(\widehat{I}_{j,\hbar}) = 0$, follows from the commutation relations

$$[\widehat{H}_\hbar, \widehat{I}_{j,\hbar}] = 0, \qquad [L_{k,\hbar}, \widehat{I}_{j,\hbar}] = [L_{k,\hbar}^\dagger, \widehat{I}_{j,\hbar}] = 0 \quad \text{for all } j, k.$$

Indeed, the Hamiltonian term vanishes, and $\widehat{I}_{j,\hbar}$ commutes with $L_{k,\hbar}^\dagger L_{k,\hbar}$, so $L_{k,\hbar}^\dagger \widehat{I}_{j,\hbar} L_{k,\hbar} - \frac{1}{2}\{L_{k,\hbar}^\dagger L_{k,\hbar}, \widehat{I}_{j,\hbar}\} = 0$.

Summary and outlook

- The quantum evolution converges in the semiclassical limit to the contact dynamics, while the dissipated quantities become genuine quantum constants of motion.
- The Lindblad dynamics exponentially damps the off-diagonal blocks $\rho_{\nu\mu}(t)$ for $\nu \neq \mu$, while leaving the diagonal blocks $\rho_{\nu\nu}(t)$ invariant: an environment-induced *superselection rule* associated with the integrals of motion.
- Dissipation does *not* destroy integrability: it is aligned with the classical invariants, leading to pure dephasing rather than relaxation. While non-generic, this situation is physically relevant in several contexts, including weak coupling to classical noise, measurement-induced decoherence, and engineered reservoirs.

Summary and outlook

- Our results suggest that contact geometry provides a natural classical language for describing the semiclassical limit of certain open quantum systems, in which decoherence selects classical invariants rather than equilibria.
- This perspective opens the door to systematic constructions of open quantum models in which dissipation, integrability, and semiclassical consistency are aligned by design, rather than imposed by ad hoc assumptions.

Thank you for your attention!

L. Colombo, A. López-Gordón, Egorov-type semiclassical limits for open quantum systems with a bi-Lindblad structure, Anal. Math. Phys. **16** (2026) 118, doi:10.1007/s13324-026-01265-9

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